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Adaptive Control Techniques for Large Space Structures

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This report summarizes the research performed ~~during the period~~ 1 June 1985 to 31 May 1986. on adaptive control techniques for Large Space Structures (LSS). The research effort concentrated on two areas: (1) on-line robust design from identified models - what is referred to here as adaptive calibration; and (2) an analysis of slow-adaptation for adaptive control of LSS. The report summarizes the results obtained in these areas and also includes Appendices which contain technical articles: (1) Adaptive Control ... and Robust Control (3) On the use of the Method of Moments for Robust Adaptive Control ... (4) Conditions for Convergence and Divergence of Recursive Adaptive Linear Systems. (jhd)

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Contents

List of Figures

Chapter 1

Introduction

1.1 Background and Motivation

The Large Space Structure (LSS) research program was originally formulated in late 1982 in response to the increasing concern that performance robustness of Air Force LSS type system would be inadequate to meet mission objectives. In particular, uncertainties in both system dynamics and disturbance spectra characterizations (both time varying and stochastic uncertainty) significantly limit the performance attainable with fixed gain, fixed architecture controls. Therefore, the use of an adaptive system, where disturbances and/or plant models are identified prior to or during control, gives systems designers more options for minimizing the risk in achieving performance objectives.

The aim of adaptive control is to implement in real-time and on-line as many as possible of the design functions now performed off-line by the control engineer; to give the controller "intelligence". To realize this aim, both a theory of stability and performance of such inherently nonlinear controls is essential as well as a technology capable of achieving the implementation.

The issues of performance sensitivity, robustness, and achievement of very high performance in an LSS system can be effectively addressed using adaptive algorithms. The need to identify modal frequencies, for example, in high-performance disturbance rejection systems has been shown in ACOSS (1981) and VCOSS (1982). The deployment of high-performance optical or RF systems may require on-line identification of critical modal parameters before full control authority can be exercised. Parameter sensitivity, manifested by performance degradation or loss of stability (poor robustness) may be effectively reduced by adaptive feedback mechanizations. Reducing the effects of on-board disturbance rejection) is particularly important for planned Air Force missions. For these cases, adaptive control mechanizations are needed to produce the three-to-five orders-of-magnitude reductions in line-of-sight jitter required by the mission.

Research is essential to identify the performance limitations of adaptive strategies for LSS control both from theoretical and hardware mechanization viewpoints. The long range goal of this research program is to establish guidelines for selecting the appropriate strategy, to evaluate performance improvements over fixed-gain mechanizations, and to examine the architecture necessary to produce a practical hardware realization. The initial thrust, however, is to continue to build a strong theoretical foundation without losing sight of the practical implementation issues.

1.2 Research Objectives

The aims of this research study are to extend and develop adaptive control theory and its application to LSS in several directions. These include:

1. **Theoretical Development:** The initial emphasis has been on slow adaptation, since this covers may LSS situations. Later on we will examine fast adaptation. The theory developed here will provide for:
   (a) estimates of robustness, i.e., stability margins vs. performance bounds;
   (b) estimates of regions of attraction and rates of parameter convergence to these regions;
   (c) extension of the present linear finite dimensional adaptive theory to include nonlinear and infinite dimensional plants and controller structures; and
   (d) extensions to decentralized systems.
2. **Parameter Adaptive Algorithms** Assess the behavior of different algorithms, including: gradient, recursive least squares, normalized least mean squares, and nonlinear observer (e.g, Extended Kalman Filter).
3. **Parametric Models:** Assess the impact of model choices. In particular we will examine the effect of explicit and implicit model choices. An explicit model, for example, is a transfer function whose coefficients are all unknown. In an implicit model transfer function, the coefficients would be functions of some other parameters. Implicit models usually arise from physical or experimental data, whereas explicit models are selected for analytical convenience.
4. **Adaptive Nonlinear Control:** Although our early effort is to study adaptive linear control, there are may LSS situations where the control is nonlinear, e.g., large angle maneuvers, slewing.

1.3 Current Status

At the present time we stand at the beginning stages of the theoretical development in adaptive control. The result of recent efforts are contained in the selected papers in the Appendix and the references therein. A summary of earlier efforts is contained in the recently published textbook *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986. This publication is an outgrowth of research supported under this contract and involved a considerable amount of collaborative effort among several researchers in the field of adaptive control. The text discusses adaptive systems from the viewpoint of stability theory. The emphasis is on methodology and basic concepts, rather than on details of adaptive algorithm. The analysis reveals common properties including causes and mechanisms for instability and the means to counteract them. Conditions for stability are presented under slow adaptation, where the method of averaging is utilized. In this latter case the stability result is *local*, i.e., the initial parametrization and input spectrum is constrained. Based on this analysis, a conceptual framework is now available to pursue the issues of slow adaptive control of LSS.

To remove the restrictiveness of slow adaptation requires an understanding of the transient behavior of adaptive systems. A preliminary investigation is reported in Kosut et al (1986). The

transient behavior of not-slow or even rapid adaptation is a significant problem in the adaptive control of LSS, e.g., rapid retargeting.

Another approach to adaptive control is to calibrate (or tune) the controller based on a current estimate of the LSS model. This involves not just knowing *one* model, but rather, a model *set*. This problem, which we refer to as *adaptive calibration*, is essentially that of developing a technique of on-line robust control design from an identified model. Although we have worked on this problem for some time it is only recently that we have established a theoretical basis for estimating model error from system identification [see Kosut (1987), a reprint is in the Appendix]. This research has raised many new questions which need to be considered, e.g., what is the appropriate robust controller parametrization; how does it relate to model parametrization; how to iterate on the data if the estimate of model error is too large; what are the heuristics for experiment design.

1.4 Selected Publications to Date

1.4.1 Journals and Conferences

R.L. Kosut, "On The Use of The Method of Averaging for the Stability Analysis of Adaptive Linear Control Systems", *Proc. IEEE CDC*, Los Angeles, CA, Dec. 1987.

R.L. Kosut, "Conditions for Convergence and Divergence of Parameter Adaptive Linear Systems", *Proc. ISCAS 1987*, Philda., PA, May, 1987.

R.L. Kosut, "Adaptive Calibration: An approach to Uncertainty Modeling and On-Line Robust Control Design", *Proc. 25th IEEE CDC*, Athens, Greece, Dec. 1986.

R.L. Kosut, "Towards an On-Line Procedure for Automated Robust Control Design: The Adaptive Calibration Problem", presented at *1986 ACC*, June 1986.

R.L. Kosut,[1] I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, and C.R. Johnson, Jr., "Transient Analysis of Adaptive Control", submitted to *IFAC 10th World Congress*, Munich, Germany, July 1987.

R.L. Kosut,[1] and R.R. Bitmead, "Fixed-Point Theorems for Stability Analysis of Adaptive Systems", *Proc. IFAC Workshop on Adaptive Systems*, Lund, Sweden, July 1986.

R.L. Kosut,[1] and R.R. Bitmead, "Linearization of Adaptive Systems: A Fixed-Point Analysis", submitted, *IEEE Trans. on Circuits and Systems; Special Issue on Adaptive Systems*, Sept. 1987.

I.M.Y. Mareels, R.R. Bitmead, M. Gevers, C.R. Johnson, Jr., R.L. Kosut,[1] and M.A. Poubelle, "How Exciting Can a Signal Really Be?", to appear, *Systems and Control Letters*.

R.L. Kosut,[1] B.D.O. Anderson, and I.M.Y. Mareels, "Stability Theory for Adaptive Systems: Methods of Averaging and Persistency of Excitation", *IEEE Trans. on Aut. Contr.*, to appear, Jan. 1987.

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., and R.L. Kosut, "Stability Theorems for the Relaxation of the SPR Condition in Hyperstable Adaptive Systems", *IEEE Trans. on Aut. Contr.*, submitted.

[1]Research performed while R.L. Kosut was a Visiting Fellow at the Australian National University.

R.L. Kosut and C.R. Johnson, Jr., "An Input-Output View of Robustness in Adaptive Control", *Automatica: Special Issue on Adaptive Control*, **20**(5):569-581, Sept. 1984.

R.L. Kosut,[2] and B. Friedlander, "Robust Adaptive Control: Conditions for Global Stability", *IEEE Trans. on Aut. Contr.*, **AC-30**(7):610-624, July 1985.

1.4.2 Books

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

R.L. Kosut, "Methods of Averaging for Adaptive Systems", *Adaptive Systems: Theory and Applications*, Editor: K.S. Narendra, Plenum Press, 1986.

R.L. Kosut and M.G. Lyons, "Issues in Control Design for Large Space Structures", *Adaptive Systems: Theory and Applications*, Editor: K.S. Narendra, Plenum Press, 1986.

R.L. Kosut, "Adaptive Control of Large Space Structures: Uncertainty Estimation and Robust Control Calibration", *Large Space Structures: Dynamics and Control*, Editors: S.N. Atluri and A.K. Amos, Springer-Verlag, 1987.

1.4.3 Other Related Publications

M.L. Workman, R.L. Kosut,[3] and Franklin, "Adaptive Proximate Time-Optimal Servomechanisms: Continuous-Time Case", *Proc. ACC*, pp.589-594, June 1987, Minneapolis, MN.

M.L. Workman, R.L. Kosut[3], and Franklin, "Adaptive Proximate Time-Optimal Servomechanisms: Discrete-Time Case", *Proc. CDC*, Dec. 1987, Los Angeles, CA.

R.L. Kosut,[4] A. Pascoal, S. Morrison, and M.L. Workman, "Time-Optimal Control of Large Space Structures", *Proc. SPIE*, Jan. 1988, Los Angeles, CA.

S. Philips, R.L. Kosut[3], and G.F. Franklin, "An Averaging Analysis of Discrete-Time Indirect Adaptive Control", to appear, *Proc. ACC*, June 1988, Atlanta, GA.

1.5 Collaborative Research Effort

Dr. Kosut has continual exchanges and collaboration with many of the leading researchers in the field of adaptive control and system identification. He has made two visits to the Australian National University in order to work jointly with Prof. B.D.O. Anderson and Prof. R.R. Bitmead of the Systems Engineering Dept. These visits were supported by AFOSR under this contract with travel grants from the Australian National University and a joint research grant with ISI and Cornell University (Prof. C.R. Johnson, Jr. was co-principal investigator) under funding from the

[2]Started under contract F4920-81-C-0051.

[3]Research supported partly by NSF Industry/ University Cooperative Research Program under Grant ECS-8605646.

[4]Research supported by SDIO/IST and managed by AFOSR.

NSF/US Australia Cooperative Research Program. Dr. Kosut also visited on several occasions with Prof. Kokotovic of the Coordinated Science Lab of the University of Illinois in Urbana, Illinois, and with Prof. C.R. Johnson, Jr. of the School of Electrical Engineering of Cornell University in Ithaca, New York. The textbook *Stability of Adaptive Systems* (MIT Press, 1986) and numerous joint papers (see Section 1.4) were a direct outcome of these visits.

Dr. Kosut has also been invited to spend some time in Sweden with Prof. K.J. Astrom of the Lund Institute of technology and with Prof. L. Ljung of Linkoping University, both of whom are experts in the field of system identification and adaptive control. These visits are supported by AFOSR, Directorate of Aerospace Sciences, the NSF Industry/University Cooperative Research Program, and the NSF U.S./Sweden Cooperative Research Program.

Prof. E. Crawley from the Dept. of Aeronautics and Astronautics of MIT will be spending his sabbatical leave, starting in December 1987, at Stanford University. There he will collaborate with Dr. Kosut on a textbook on the dynamics and control of large space structures.

Dr. Kosut is also collaborating with Professor C. R. Johnson, Jr. from Cornell University on a textbook on robust and adaptive control.

1.6 Future Directions

Based on our recent results as reported here, we envision near-term activity in several directions, including:

- transient analysis of adaptive control;
- analysis of adaptive calibration;
- *decentralized control structures;*
- effect of nonlinear and infinite dimensional phenomena;
- effect of different algorithms and parametrizations.

1.7 Organization of Report

In Chapter 2 we provide an overview of our rerseach efforts in this past year. Specifically, the chapter contains brief discussions of some of the research areas listed above, pointing out the general research directions, and in particular, their specific applicability to LSS systems.

Chapter 3 contains a parallel, but more in-depth technical discussion of the same research topics.

There is also an Appendix which contains some of our recent publications from this past year.

Chapter 2

Overview of Research Activities

2.1 Motivation and Objectives

Many of the envisioned future large space structure (LSS) missions will impose stringent performance demands on tracking accuracy and structural vibration attenuation. For example, some planned missions will require three to five orders of magnitude reduction in line-of-sight optical jitter. Both active feedback control and passive damping will thus be a practical necessity, and moreover, their design will require a model of the LSS system whose accuracy is compatible with the performance demands. Even random variations in materials and manufacturing tolerances will significantly degrade closed-loop performance. For example, the need to have very accurate models for high performance disturbance rejection has been demonstrated in the ACOSS(1981) and VCOSS(1982) programs. The deployment of high-performance optical or RF systems will also require accurate models before full control authority can be exercised. As a result of these and other research programs, it is clear that the on-orbit dynamics of LSS will not be sufficiently like those obtained from either ground-testing or even from sophisticated computer generated modeling techniques, such as finite element modeling. Current structural modeling techniques are just not sufficiently accurate or able to account for all the possible sources of parameter variation. Therefore, under these conditions, it will be necessary to identify the LSS dynamics directly from on-orbit measurements, and simultaneously, tune or re-design the control. Hence, the control design cycle will be an *adaptive* process, typically starting with a nominal low-performance design based on a coarse model, and then re-designed from on-orbit data. The adaptation may take place either off-line or simultaneously while the system is operating, but in either case, the process will require on-orbit data.

As an illustration of adaptive control for an LSS system, consider the two-level control architecture depicted in Figure 2.1. The two levels consist of a colocated rate damping controller and a non-colocated high performance controller. The colocated controller consists of active and/or passive rate damping devices placed at critical structural locations, and their design requires only a coarse knowledge of system dynamics. This is an inherently robust controller, but yields low performance, and has been referred to as the low authority control (LAC) system. The high performance controller is non-colocated and requires an accurate knowlege of critical modes, and hence, is very sensitive to structural parameter variations. This controller, referred to as the high authority control (HAC) system, provides high damping and mode shape adjustment in selected modes in order to meet the performance demands. We remark that the stringent performance demands cannot be

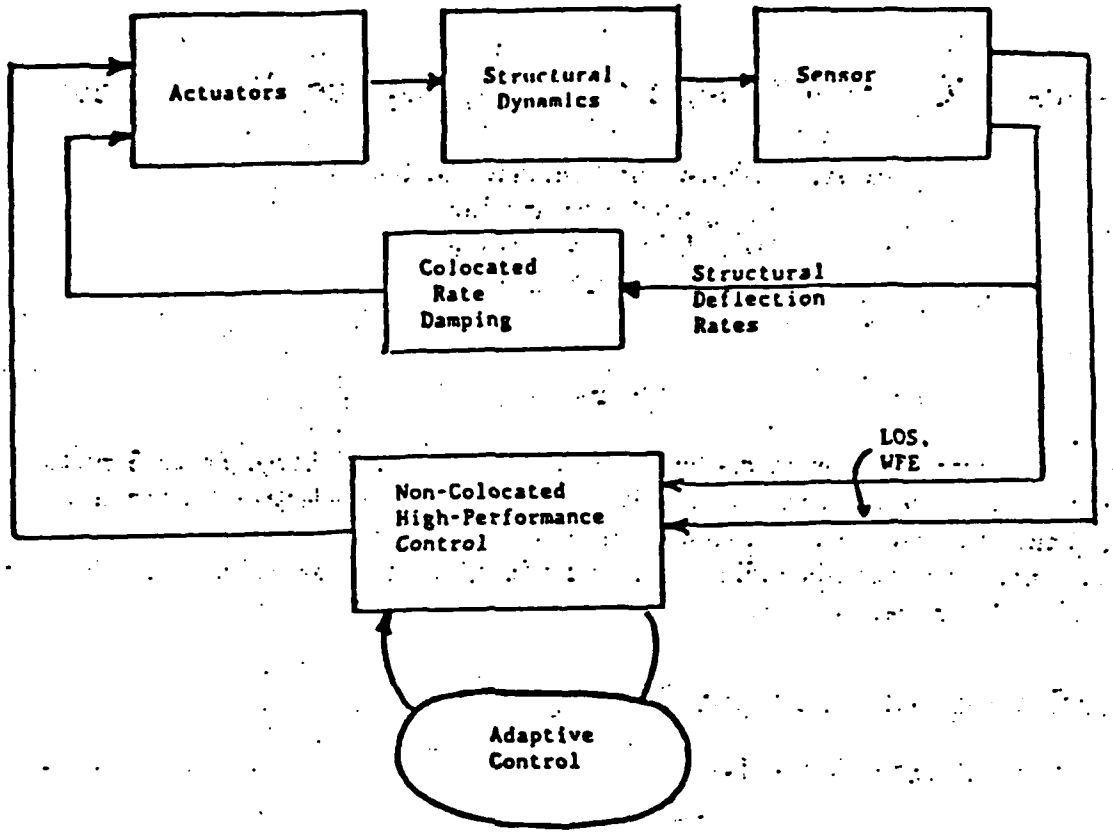

Figure 2.1: Two-Level Control Architecture

met with passive or active rate damping mechanisms alone. Also, the synthesis and design of the HAC/LAC system has to be properly integrated to avoid performance degradation due to modeling inaccuracies, see, e.g., Aubrun et al.(1979). With this architecture only the HAC-system is likely to be tuned by an adaptive system, as shown in Figure 2.1.

The limitations of the HAC and LAC systems performance with respect to structural parameter variations is illustrated in Figure 2.2. Observe that the LAC system provides significant robustness to parameter variations, but the performance increase over the open loop structure is moderate. The opposite is true for the HAC system, which provides significant performance increase, but is very sensitive to parameter variations. The adaptive control of the HAC system allows for a much wider latitude in parameter variation while maintaining the performance level required.

The objective of our research program is to establish the theoretical foundations and performance limitations for the application of adaptive control to large space structures. To realize this aim, theories of both stability and performance of such an inherently nonlinear control system are essential, as well as a technology capable of achieving the implementation.

2.2 Summary of Recent Research

2.2.1 Stability and Convergence Analysis

Until recently, the underlying conditions for stability and convergence and the causes for divergence and instability in adaptive control systems have not been entirely understood. We our proud to

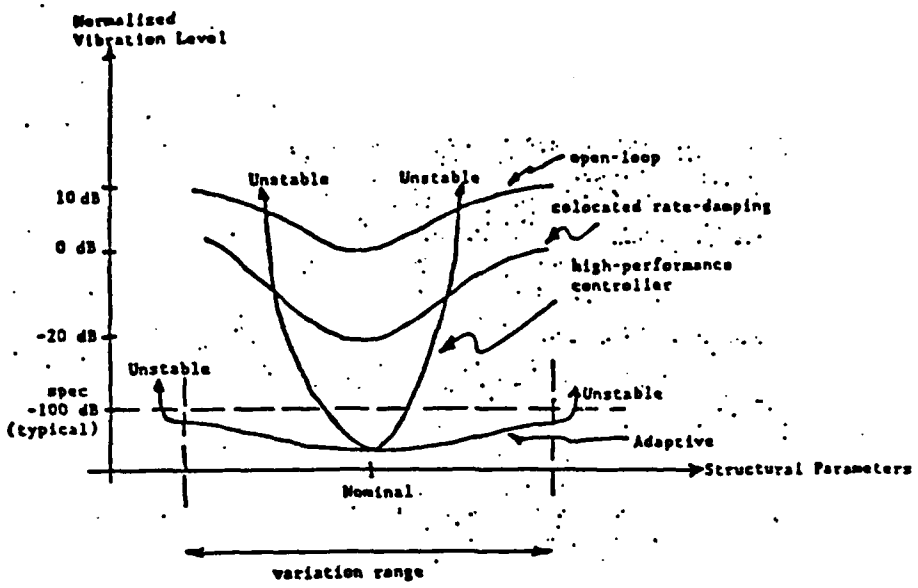

Figure 2.2: HAC/LAC Performance vs. Structural Parameter Variation

say that our recent research, supported by the AFOSR Directorate of Aerospace Sciences, has contributed significantly to a deeper understanding of these phenomena. In fact, when we began our research program, the then existing theory of adaptive control was wholly inapplicable, not only for LSS systems, but for any dynamic system. The main reason for this was that the then existing theory required perfectly accurate models, which of course is an unrealistic requirement.

As a consequence of this lack of appropriate theory, a major emphasis in our recent research has been the development of a fundamental theory of adaptive control which is applicable to realistic dynamic system models.

In commencing our research program, we first recognized that the problems in establishing realistic conditions for stability and convergence arise not only from the unique dynamical character of LSS systems, but also from the basic philosophy underlying the design of an adaptive control system. As depicted in Figure 2.3, there are essentially two basic processes, namely: (1) a model estimator, and (2) a control design rule.

The model estimator operates on the input-output data obtained from measurements of the system to be controlled, producing a model estimate from a pre-determined model set. The model estimate is transformed by the control design rule into a controller to be used in feedback with the actual system. The fundamental question is: when will it work? That is, under what conditions will the model estimate converge to a good model in the allowed model set. By a good model is meant one that produces a controller, via the control design rule, which when applied to the actual system yields acceptable closed-loop performance. Even if such a good model exists, the estimated model may not converge, even if the estimated model is initialized close to a good model. Moreover, if convergence is too slow, then unacceptable behavior can occur during the learning process. Thus, convergence alone is not a sufficient condition for establishing good performance. We also need information about the convergence rate, particularly how it can be affected by user choices of inputs, data filters, etc.

During our research on the problem of stability and convergence we worked closely with several other researchers. A summary of these efforts is contained in the recently published textbook *Sta-*

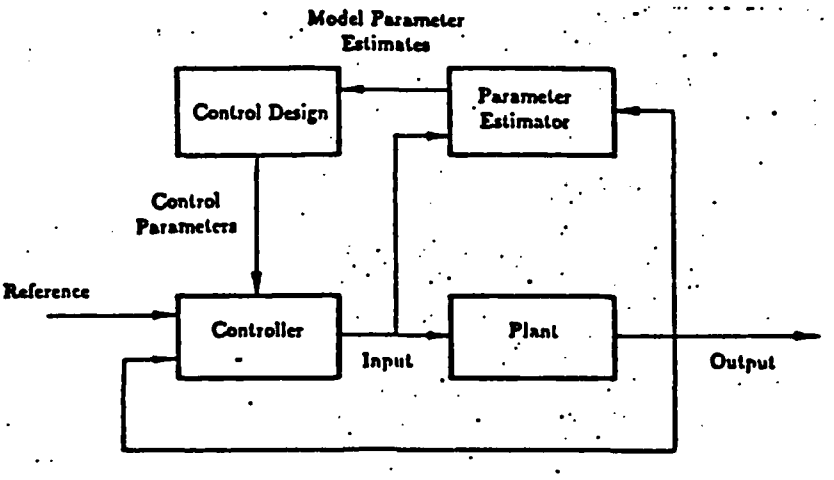

Figure 2.3: Adaptive Control

bility of Adaptive Systems: Passivity and Averaging Analysis (MIT Press) by Anderson et al.(1986). The stability analysis focuses on slow adaptation and the use of some of the classical methods for analyzing differential equations, e.g., linearization, the method of averaging, and Lyapunov's second method, see, e.g., Hale(1969). The material in the text represents modifications and refinements of earlier work, specifically: Astrom (1983, 1984) showing how the method of averaging explains instabilities and drift; Reidle and Kokotovic (1985, 1986) on slow adaptation and the integral manifold; Kosut, Anderson, and Mareels(1987) on the relation between averaging and persistent excitation; Kosut (1985) and Bodson et al.(1986) on nonlinear averaging analysis and determining the rate of convergence; and Kosut and Anderson (1986), Kosut and Johnson(1984) on linearization and local stability.

The above analysis reveals common properties including the causes and mechanisms for instabilities and the means to counteract them. In particular, conditions for stability are presented under slow adaptation, involving the method of averaging. Based on this analysis, a conceptual framework is now available, for the first time, to pursue adaptive control of LSS systems, under slow adaptation. We remark that in the literature on adaptive control, this issue is often referred to as the *robustness* of adaptive control, i.e., what happens under non-ideal conditions such as the effect of unmodeled dynamics and disturbances. [An historical perspective of this research area is provided in Section 2.7.1].

To remove the restrictiveness of slow adaptation requires an understanding of the transient behavior of adaptive systems. Preliminary investigations are reported in Kosut and Bitmead(1986) and Kosut et al.(1987). This pursuit involves new analysis methods, not necessarily averaging, and further development of these and other techniques is one of the our research goals.

2.2.2 Uncertainty Estimation

Referring again to Figure 2.3, we see from the previous discussion that the adaptive control system is working if one can prove that the estimated model converges to a "good" model of the true

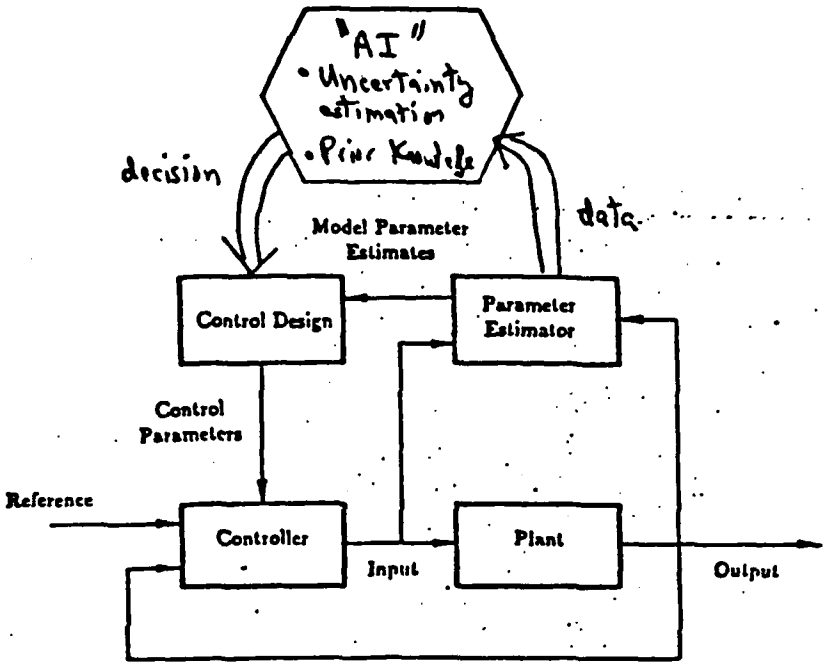

Figure 2.4: Adaptive Control with AI Outer-Loop

system. And, for example, analysis techniques based on the method of averaging, as described above, do provide an assessment of the adaptive system under slow adaptation.

Another route, which we are currently exploring, is to modify the identification process so that it produces not only an estimated model, but a measure of model uncertainty as well. With such an estimate in hand the control design rule can be modified to account for the model uncertainty. If the resulting controller is too cautious, because the model estimate is too uncertain, then it is necessary to repeat the estimation procedure so as to obtain a better model.

Thus, the adaptive control now contains an outer loop, as shown in Figure 2.4 which provides some expert advice as to when to use the model estimate. Other "artificial intelligence" (AI) features can also be incorporated in the outer loop. This feature will be expanded upon in our new program.

In our current research program we are concentrating on a particular procedure for obtaining a measure of model uncertainty for linear time invariant plants. The process is referred to as *uncertainty estimation* and the outcome is a frequency domain expression for the model uncertainty. The motivation arises from robust control theory which utilizes frequency domain expressions for characterizing a *set of uncertainty* within which lies the true plant, e.g., Francis and Zames(1983), Safonov et al.(1981), Doyle and Chu(1985), Vidyasagar(1985). The set of uncertainty is extracted from input-output measurements. Our current approach uses a combination of parametric prediction error methods together with standard non-parametric (spectral) estimation methods. The set of uncertainty produced by the modified identifier allows for performance evaluation *before* the controller is adjusted. If the predicted performance is not satisfactory then the identification process is repeated under different conditions which will reduce model uncertainty where needed. The resulting controller is designed to be robust with respect to the estimated set of uncertainty. Thus, a large set of uncertainty requires a cautious or low authority controller, whereas a small set of uncertainty will result in a high authority controller. Also, the order of the controller can vary with the set of uncertainty. Details have been reported in Kosut(1986,1987a,b).

2.3 New Research Directions

The previous discussion emphasized two important areas of basic research, namely: (i) methods of stability and convergence analysis, and (ii) uncertainty estimation. In order to facilitate our research in these areas we have concentrated on the simplest of model parametrizations, namely, finite dimensional transfer functions where the adaptive parameters are the coefficients. This choice leads to the standard forms of recursive parameter adaptive algorithms for identification and adaptive control, e.g., gradient, recursive-least-squares, etc. Despite this simple model choice, we are nonetheless still faced with basic unresolved research issues in both of the mentioned areas. At the same time, we feel that there is now a sufficient foundation to extend the previous results and develop new theory and methodology applicable to other model parametrizations and other types of controllers. For example, *physical* model parametrizations, typical of LSS characteristics, include: continuum models, wave models, and finite element models. These types of models will require *new* identification and adaptive control algorithms, and will stretch and test our previously developed methods for stability and convergence analysis. Other control types include: *nonlinear* controllers such as might arise from large angle maneuvers or rapid-retargetting and tracking, and decentralized controllers which arise from restricted information distribution, i.e., decentralized information patterns.

Many new unanswered questions arise from these choices, specifically how they affect system identification and adaptive control. For example:

- What is the best procedure for developing parameter estimation algorithms from physical model parametrizations, e.g., gradient, Gauss-Newton, etc?
- How do we use an estimated physical model for control design?
- Will our present analysis tools for stability and convergence be valid for these model choices? What are the necessary modifications?
- How well do such model forms approximate the "true" LSS system? How well do they have to, particularly when the intended use is control design? What is the effect on model structure selection of the control design criteria, such as vibration suppression, rapid tracking, decentralized information, etc?
- What are the best test conditions for the identification experiment when the intended use of the estimated model is for control design? How does one choose the experiment design variables, such as test signals, sampling rate, data filters, and so on?
- What is the affect of utilizing the estimated model for nonlinear control?
- How does a decentralized control architecture modify system identification? Under what conditions will a truly decentralized adaptive control system converge?
- Given any model, is it better, in general,to directly estimate the model or to directly adjust the controller parameters based on the model?

These questions broadly cover the basic research areas. However, it is often difficult to discuss one area without another.

2.4 Model Structure Selection for Control Design

Consider again the adaptive control system depicted in Figure 2.3. The controller parameters are adjusted in accordance with estimated model parameters obtained from on-line data. The first step then, in designing the adaptive control system, is to select a model structure or parametrization.[1]

2.4.1 Model Parametrization

We will restrict our discussion here to parametric transfer function models. There are two basic choices for parameters:

- physical parameters
- canonical parameters

Physical parameters include masses, spring constants, electrical circuit quantities, etc. Canonical parameters are best exemplified by transfer function coefficients, which usually depend in a complicated manner on the physical parameters. Moreover, if the transfer function represents a sampled-data system, then the relation between the physical parameters and the transfer function coefficients becomes even more complicated, particularly when the sampling rate is low.

In parameter estimation, the above parametrizations offer different adavantages and disadvantages. The estimation of transfer function coefficients can be very direct, because it is possible to express the prediction error as a linear function of these parameters, thus leading to simple estimation algorithms, such as recursive least squares. Physical parameters are directly meaningful, but because they enter in a complicated manner into the transfer function coefficients, the prediction error is also a complicated function of the physical parameters, making direct estimation schemes difficult to design. Often a system model can be characterized by a few physical parameters, whereas the transfer function may consist of very many coefficients.

One can now ask the question: which of the above model parametrizations is "better"? To answer this question, we first remark that *neither* is correct, because the true system is different than either model. Even the so called physical transfer function model is undoubtedly derived under certain assumptions, such as small deflections, linear elasticity, uniform mass density, etc. Hence, the choice should be made by considering the intended use of the model.

2.4.2 Model Evaluation

In control design, which is the case we consider here, what is needed is an estimate of the transfer function, whose accuracy is determined by the *closed-loop* performance. In this regard the parametrization is a convenient way to obtain this result. However, the different parametrizations invoke parameter estimation algorithms which may have significantly different convergence properties and numerical stability properites. Hence, although closed-loop performance is the ultimate aim, other criteria must be considered such as numerin its complexity, e.g., there should not be too many parameters, but just enough for the intended use of the model.

[1] It is possible to parametrize the closed-loop system in terms of controller parameters, which are then directly adjusted, rather than going through an indirect route via model parameter estimation. An example is the model reference adaptive control.

In summary, to evaluate a particular model parametrization for on-line parameter estimation, when the intended use of the estimate is for control design, the analyst needs to determine the relations among the following choices:

- a representative "true" system
- a parametric model set
- a model estimator or adaptive algorithm
- a set of closed-loop performance criteria
- a control design approach, rule, or algorithm

At first glance it would appear that these choices cannot be separately analyzed. This is not the case at all! A very clear presentation of parametric transfer function estimation is contained in the recently published textbook by Ljung(1987). [A brief summary is provided in the Appendix in Section 3.4.] There we find the following important results:

- Least squares parameter estimation is equivalent, asymptotically in the length of the data record, to minimization of a weighted frequency domain criterion which penalizes the quadratic error between the true system transfer function and the parametric model transfer function. The weighting function depends on various user choices, such as the input spectrum, data filters, and the noise or disturbance model.
- The mean-square-error (MSE) between the true system transfer function and the ideal[2] estimated transfer function can be explicitly evaluated and depends on the user choices listed above.

One of the most significant aspects of the above results is that the MSE btween a particular choice of model parametrization and the true system can be evaluated *without specifying how the estimate is computed or whether it converges.* The convergence question involves the analysis techniques reported in Anderson et al.(1986). Hence, conditions for convergence add further constraints on the user choices. It is important to emphasize that since the above results depend on the choice of a "true" representaive system, they can be used for analysis and design of the estimator and estimation experiment.

Many research efforts have been undertaken to evaluate methods of system identification for distributed parameter and LSS type systems. For example, we cite Goodson and Polis(1974), Banks, Crowley, and Kunisch(1983), Rafajlowicz(1983), Schaechter(1982,1986), and Denman et al.(1986), to name a few. The difference between these reported results and what we are investigating is the connection we make between the criteria for system identification and the intended use of the estimated model for control design. To our knowledge there has never been an organized effort along the lines we suggest. It is our contention that by proceding in this organized manner, backed up by the strong theoretical foundations in Ljung(1987) and in Anderson et al.(1986), that results can be obtained which will be of scientific value in themselves, and will significantly increase our prospectives for practical system identification and adaptive control of LSS.

[2]The ideal estimate minimizes the least squres criterion for a specified length of the data record. This is *not* the estimate which is recursively computed from on-line data. These two estimates are, at best, asymptotically equivalent, with the recursive estimate having a larger variance, the size of which is dependent on the specific algorithm for on-line estimation.

2.4.3 Use of the MSE for Analysis

The MSE for model error is a function of the true plant, the estimation criteria, and the length of the data record. Hence, computation of the MSE provides an *analysis* tool for determining the effectiveness of on-line transfer function estimation. We remark again for emphasis that the MSE is *not* dependent on *how* the estimate is computed. The parametrization alone can be assessed for control design.

We are continuing to examine the effect of a number of model parametrizations on the MSE corresponding to both parametric and non-parametric methods of transfer function estimation. Specific studies will include least-squares and least-peak methods of parameter estimation, with parametric transfer function representations such as might arise from continuum modeling, wave modeling, or finite element modeling of an LSS. The non-parametric transfer function estimation methods will be based on spectral estimation techniques. The effect of these choices on estimation critera and the MSE will be examined, specifically where the intended use of the model is for on-line control design. Computing the MSE will be a significant part of this effort.

Continuum Modeling As one example of the kind of analysis that might be accomplished, we will examine the use of simple continuum models, such as beams, shells, plates, etc., with parametrized spatially varying parameters. For example, a true system can be represented as a non-uniform elastic rod, whereas the model set is a uniform rod with the parameters to be estimated being the constant tortional mass and stiffness. In this case we will try to determine what amount of distributed tortional mass and stiffness can be sufficiently well approximmated by a uniform mass and stiffness for the purposes of control design. We will repeat the analysis with the same true system, but with other parametric models from the above list as brifly described below.

Wave Modeling Wave models may be quite useful for designing active controllers which act like mechanical energy absorbers, see, e.g., von Flotow(1984), Hagedorn(1985). The wave model is parametrized in terms of the frequency dependent "scattering coefficients" which describe the wave properties.

Finite Element Modeling The mass and stiffness matrices which arise from a finite element analysis are dependent on a variety of physical parameters, such as areas, radii, mass densities, etc. Small variations about nominal values can be regarded as the model parameters, see, e.g., McIntosh and Floyd(1985).

Distributed Spatial Measurements The same methodology permits evaluating the benefits of distributed spatial measurements, or what is more likely, many closely spaced measurements. The availability of such measurements can significantly simplify the physical parametrization of transfer function models. We remark that the spatial measurements might not be just position and velocity but might involve the outputs of very many strain gauges placed appropriately on the structure.

2.4.4 Experiment Design

The MSE can be used to aid in designing the estimation experiment. The objective being that during the *actual* experiment the maximum amount of data is extracted in computing the transfer

function estimate. This particular issue, referred to as *experiment design*, has been extensively investigated, e.g., Goodwin and Payne(1977). Recently, there has been a specific interest in using these results in order to design experiments which optimize the use of the transfer function estimate in control design, e.g., Ljung(1985), Wahlberg and Ljung(1986), Gevers and Ljung(1986). These results minimize a weighted norm of the MSE with respect to free parameters or choices in the experiment, e.g., input spectrum, model order, etc. Since the MSE depends on the true plant, the methods offer usefull guidelines for designing the experiment. The basic idea is to make the criterion for estimation similar to the criterion for control design as listed under Task 1 in Section 2.3.

At the present time the use of the MSE as a measure of experiment design for the case when the intended use of the transfer function estimate is control synthesis is still in the beginning stages. One of the major problems is that measures of closed loop performance are complicated functions of the MSE and the criterion for estimation.

We have been studying this relationship with particular emphasis on the LSS dynamic representations previously mentioned and the control design appraoches discussed next.

2.4.5 Control Design Criteria and Approaches

In the previous discussion we have emphasized the need to evaluate the model structure for identification in terms of the intended use of the estimated model for control design. In this section we briefly discuss control design criteria and approaches. The control design criteria typically include:

- vibration suppression
- regulation
- tracking

Typical constraints include:

- robustness to model error
- actuator limitations
- decentralized architectures

Control design methods for achieving these goals are hard to classify or codify, but a partial list is as follows:

- pole placement
- minimum variance
- LQG
- stable factorization
- H_∞-optimization

First of all, it is not possible here to discuss all these methods. Secondly, what is of primary interest is the goal of the methodology and how it is reflected in the criterion for system identification. Now, although it is not easily apparent, the above design approaches can all be subsumed using the so-called stable factorization (SF) approach as presented in Desoer et al.(1980) and Vidyasagar(1985).

Stable Factorization Approach The basic idea is to "factor" the plant into a ratio of two stable transfer functions, or operators, in the more general case. It can be shown that these factors allow for the parametrization af *all stabilizing controllers* in terms of a single stable transfer function, referred to as the *Youla parameter* [Youla et al., 1976]. Moreover, and this is perhaps the most significant aspect of this controller representation, *all the closed-loop transfer functions from exogenous inputs to internal outputs depend linearly on the Youla parameter.* The use of this approach specifically for vibration control of lightly damped systems and LSS can be found in Bennett et al.(1987), Helmicki et al.(1987), and Pichet(1985).

To relate the SF approach to pole placement simply means to select the stable factors so that the poles are placed where desired in the factors.

The LQG approach requires the introduction of a norm in the signal space, i.e., average integral squared in time. This translates to what is called the H_2 norm of the closed-loop transfer function matrix, i.e., integral squared in the frequency domain. In fact, the LQG state space controller-observer solution is equvalent to a particular stable factorization, see, e.g., Nett et al.(1984).

Robustness The SF approach also provides for a natural measure of robustness to model error. It turns out that the weakest topology for handling robustness is exactly a ratio of stable factors, Vidyasagar(1984). Thus, robustness to model error is more naturally seen as perturbations in the stable factors. The appropriate norm is the peak value, as a function of frequency, of the closed-loop transfer function, referred to as the H_∞-norm. Techniques for H_∞-optimization are attempting to find that control which achieves optimal robustness to model error, see, e.g., Francis and Zames(1983), Boyd et al.(1987).

Preliminary analysis of robustness to model error in the LSS environment has been examined in Kosut et al.(1983), Bhaya and Desoer(1985), to name a few. We have examined the use of SF approaches and how to affect the identification criterion so as to obtain an estimate which results in a control design that is maximally robust.

2.5 Computing the Model Estimate from Data

In general, adaptive algorithms share a similar genesis and purpose, namely to adjust a parameter vector to asymptotically approach a member of the set which minimizes an average squared error, such as prediction error for system identification or tracking error for direct adaptive control.

In the ideal case, the parametrization can be selected so as to achieve what is called *perfect matching*, i.e., for some restricted class of exogenous inputs (usually no disturbances), and a sufficient number of parameters (no unmodeled dynamics), there is a single minimum which produces either zero error or the error is zero-mean white noise.

Under realistic, but non-ideal, conditions, the best that can happen is that the adaptive pa-

rameters asymptotically approaches a small neighborhood of a local minima of the average squared error, see, e.g., Anderson et al.(1986).

2.5.1 Adaptive Algorithms

Parameter Adaptation One method to achieve this result is to adjust the parameters in the negative gradient direction of the instantaneous squared error, a procedure that was once referred to as the "MIT-Rule" [see Section 2.7.1]. More general procedures involve adjustments in accordance with Gauss-Newton iterations, thereby taking into account second order effects. The recursive least square algorithm is the classic example of this approach. These methods can be extended for a host of error measures, which are not required to be quadratic, see, e.g., Ljung and Sodestrom(1983), Goodwin and Sin(1984).

Constructing the Error Gradient from Measured Signals A difficulty with constructing the above algorithms is that the error gradient with respect to the parameters, as if they were fixed, is required. However, in most cases, the error gradient is a function of the true, but unknown plant. The actual algorithm must then use an approximation, which is loosely referred to as the *regressor.* (An example of constructing the regressor for a simple continuum model is provided in the Appendix in Section 3.7.3.)

Algorithm Modifications: "AI" Features To insure that adaptive algorithms are working, one often includes various "safety nets" or AI features. For example, in uncertainty estimation, the controller is not changed until some level of confidence is established. Other AI features include monitoring of information, covariance monitoring, perisitent excitation conditioning, and parameter projection, to name a few. Various normalizations of the measured sugnals also enhance the ability of adaptive systems to insure bounded parameter values. These latter approaches need prior information regarding model error and parameter ranges.

On-Line Control Design Rules Once the estimated model is obtained, the control design rule (see Figure 2.3) transforms the estimate to a controller which is then implemented with the actual system. If the model estimate includes an uncertainty estimate as well, then the control design rule will involve a robust design procedure such as outlined in our discussion of the stable factorization approach.

We will continue to develop and evaluate the control design rule based on the model structure selection and the control design criteria listed before. Once the design rule is established, we need to then turn our attention to:

- experiment design
- stability and connvergence analysis

These issues will be discussed in the following sections. The experiment design problem involves computing the MSE. The stability and convergence analysis involves the method of averaging and other extentions and new techniques as required.

2.5.2 Performance of the Algorithm

Despite the above approximation, from a practical point of view it is certainly acceptable that the parameters approach and remain in a *small neighborhood of a local minimum of the average* squared error, provided that members of this set also produce acceptable performance.

We refer to each member of this set as a *tuned parameter*, and to the corresponding feedback system as the *tuned system* [see, e.g., Kosut and Friedlander(1985)]. Clearly the tuned system is the adaptive feedback system of Figure 2.3 with the adaptive parameters held fixed at a tuned setting. We can now pose the following questions regarding the adaptive system :

1. How do the tuned parameters depend on the exogenous inputs?
2. Is the adaptive system stable in a neighborhood of the tuned system. Furthermore, how small is this neighborhood?
3. What is the region of attraction to a small neighborhood of the tuned system.
4. What *is the rate of convergence to this small neighborhood.*

These questions can only be answered by application of methods of stability and convergence analysis. (See Sections 2.7 and 3.8.)

An important part of our research effort will be in developing suitable approximations to the error gradient. The suitability of a particular approximation will of course be dependent upon the results of the stability and convergence analysis.

2.5.3 Multiple Model Adaptive Control

We are also examining an adaptive control scheme which involves the selection of a fixed controller out of a finite collection, where each controller is robustly designed to account for the true plant being in a not necessarily small set of uncertainty. There are no adaptive "parameters" in the conventional sense where adjustments are made to either model or controller parameters. The parameters here involve pseudo-probabilities or weights assigned to each of the finite controllers. These quantities are computed recursively from the measured data. Various adaptive selection mechanisms which depend on these pseudo-probabilities will be developed as appropriate.

Figure 2.5 depicts the general adaptive set-up that we consider. The design of each controller $C_i(z)$ is not an adaptive control design, but rather, a robust control design task. The signals used to adaptively select the controller are the innovations sequences, or prediction errors, denoted by $\varepsilon_1(t),\ldots,\varepsilon_N(t)$, where each is obtained from the Kalman filters denoted $KF_1,\ldots,KF_N$.

Our main task is to explain how the best controller in the set of controllers can be selected to control the plant. This type of adaptive controller is not like the conventional ones where either model or controller parameters are directly adjusted. Here the parameters are contained in the mechanism for switching amongst the preselected robust controllers, which is essentially a *gain scheduling* procedure, but is adaptive in the sense that the schedule is being learned from the measured data. The gain schedule is usually set in advance. For example, in a flight control system the gain schedule is a predetermined function of the Mach number and aerodynamic pressure. One of the interesting possible advantages of this method is that although the plant may have a

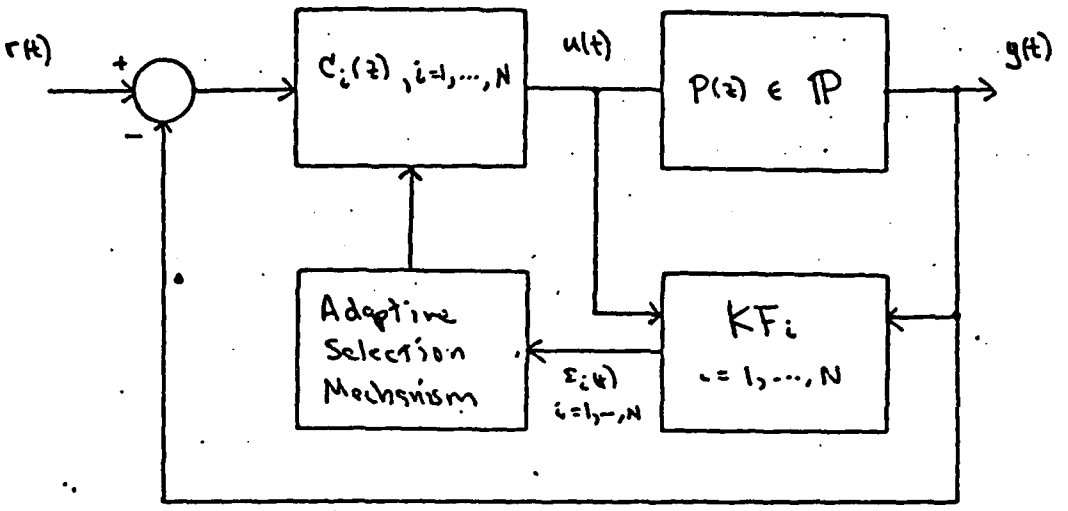

Figure 2.5: Multiple Model Adaptive Control

large number of uncertain parameters, it is possible that only a few controllers are required, and hence only a few parameters in the selection mechanism. Also the individual robust controllers can be based on uncertainty in physical parameters rather than canonical parameters, such as transfer function coefficients, as used in the conventional parameter estimation techniques, e.g., least squares with a linear regression model.

The adaptive selection mechanism as described above has been investigated in Anderson and Moore(1979) and Athans et al.(1977). In this latter reference this scheme is referred to as a Multiple Model Adaptive Control (MMAC), but the individual controllers are not selected to be robust in the manner described above. We are developing an analysis of the convergence properties of the adaptive selection algorithm following the analysis in Anderson and Moore(1979).

2.6 Computing Model Uncertainty from Data

The requisite information for robust feedback stabilization is typically a *nominal model of* the plant together with an uncertainty profile. From the above discussion, computation of the MSE for model error allows one to evaluate a control design which is based on a nominal (estimated) model together with an *a priori* estimate of model accuracy. We will use both parametric and non-parametric methods of transfer function estimation, where the non-parametric methods yield direct estimates of the accuracy. Essentially, this approach provides a direct estimate of the MSE. The accuracy to which the estimate is set depends on the criteria for control design.

The reason that we consider non-parametric methods is that they can provide estimates of model accuracy as a function of frequency, particularly over those frequency ranges where either the model structure is poorly known, or else the model accuracy in that range is unimportant for control design. The feasibility of using spectral techniques for the estimation of model error can be found in Kosut (1986,1987a), and for LSS application in Kosut(1987b).

2.7 Methods of Stability and Convergence Analysis

In order to more fully appreciate the significance of the convergence and stability problem in adaptive control we offer the following historical digression. This will also serve to motivate and place our previous results and proposed new research directions in this area.

2.7.1 A Bit of History

Over the years, the application of adaptive control theory, as reported in the literature, has not met with a uniform success. Surprisingly, some early applications to flight control were reported to be quite successful despite the lack of theory, e.g., Whitaker(1959) and Osborne et al.(1961). In more recent years, with more theory available, some reported applications–again to flight control–were not deemed successful, e.g., Bryson(1977). At the same time as these mixed results were being obtained in the area of flight control applications, several researchers were reporting good results with applications in process control and in ship steering, e.g., Astrom et al.(1965, 1973). Why these mixed results?

The answer lies partly in the engineering aspects which motivate the use of adaptive control, and partly in the overall research goals in developing an adaptive control theory. These are sometimes in conflict. The engineering application calls for a control structure which takes into account the nature of the *physical* problem. The theory is looking to solve a *canonical* problem, and hence, often fails to account for a priori, or experiential "rule-based" knowledge. A good example is provided by a brief history of the adaptive control technique developed by Whitaker(1959), later referred to as the "M.I.T.-Rule". The basic idea is to adjust the adaptive control parameters in accordance with an instantaneous gradient descent of the squared error signal as described in Section 3.4.

Although the resulting algorithm is simple to construct, and incidently worked well enough in simulations, there was no guarantee the resulting adaptive system would be stable. Probably because the causes of these difficulties were not readily forthcoming, and perhaps also there was (and still is) the desire to construct a "universely" stabilizing adaptive control, other approaches to adaptive control were explored.

One such avenue was announced in a paper by Parks(1966), where the M.I.T.-Rule was "redesigned" in such a way that the resulting adaptively controlled system had a *guaranteed* stability property. The idea was to make a certain transfer function *strictly positive real* (SPR) which has the effect of insuring that the parameter adjustment is always in the right direction. This "SPR-Rule" idea was developed later for more general situations, e.g., Monopoli(1974), Egardt(1979), and Narendra, Lin, and Valavani(1980). Further extentions and expositions along the SPR line can be found in Landau(1979), Goodwin and Sin(1983), and Kosut and Friedlander(1985).

Unfortunately, these SPR-Rule algorithms have exactly the same stability problems as the MIT-Rule algorithms, but only when unmodeled dynamics and disturbances are taken into account. This issue was perhaps not fully realized until the appearance of the work of Rohrs et al.(1982), which vividly demonstrated the potentially local unstable nature of adaptive systems by posing a "counter-example" representative of a non-ideal, but practical, circumstance. The adaptive parameters exhibited the characteristic rapid transient followed by a steady parameter drift. In this case, however, the parameters did not settle down in the constant parameter stability set. Thus, once the parameters drifted outside the stability set, the states of the controlled system became exceedingly large.

Cause of Parameter Drift Instability In scanning the literature, the earliest reference which contains a rigorous analysis of this instability phenomena appears to be in a note by James(1971). A Floquet analysis is applied to an MIT-Rule one-parameter linear adaptive system (adaptive feedforward only) with periodic inputs. The result is a complex stability-instability boundary revealing multiple resonance phenomena, precisely like that associated with the Mathieu equation, e.g., Hale(1969). No nonlinear analysis was undertaken for the case with adaptive feedback.

The work of Ljung(1977) which utilizes a technique of stochastic averaging can perhaps be said to herald the beginning of a nonlinear analysis technique, but its applicability to adaptive systems in non-ideal situations was not fully explored at that time. Lyapunov techniques, explored by Anderson and Johnstone(1983) and Ioannou and Kokotovic(1983), showed that *persistently exciting* signals are required to provide a uniform asymptotic stability of the ideal adaptive system, and this in turn provides robustness to various unmodeled dynamics and disturbances. Input-output formulations providing a local stability were developed in Kosut and Johnson(1984) and Kosut and Anderson(1986); these also needed persistent excitation. In none of these examinations of the stability properties of adaptive control systems was the *precise* mechanism for the parameter drift instability identified, particularly for the counter-example posed by Rohrs et al.(1982).

Method of Averaging The first insights came in a series of papers by Astrom(1983,1984), which provided an analysis based on the classical method of averaging of Bogoliubov and Mitropolski(1961). The method requires slow adaptation and/or small signal magnitudes, and under these conditions Astrom was able to show heuristically the instability mechanism. This was made precise in a paper by Riedle and Kokotovic(1984,1985), where they established an *average SPR condition* which provides a sharp stability-instability boundary, again in the case of slow adaptation with periodic inputs, but only for the linearized adaptive system. The average SPR condition is significantly less restrictive than the usual SPR condition because the latter requires that a certain closed-loop transfer function be positive at *all* frequencies, whereas the former depends also on a signal spectrum and requires that the energy at those frequencies where the transfer is positive should dominate those frequencies where it is negative. This result put teeth into an obvious engineering folk theorem. Extentions of this important result to other than periodic signals and the relation to persistent excitation is provided in Kosut, Anderson, and Mareels(1986). Extending the method of averaging to the full adaptive feedback case has also been developed, e.g., Riedle and Kokotovic(1986), Kosut(1986), and Bodsen et al.(1986).

In the light of the new averaging results it is now possible to more fully understand the difficulties with the MIT-Rule. It is clear what causes the instabilities, and most importantly, how to avoid them. The MIT-Rule is "re-visited" with these new tools in hand in Mareels et al.(1986) and Mareels(1986), and results are obtained which essentially extend the results of James(1971).

2.7.2 New Research Directions

The conclusion from this bit of history, particularly for adaptive control of LSS is this: the parameters for adaptation can correspond to physically meaningful parameters, rather than canonical parameters such as transfer function coefficients. Once having selected the critical parameters, the parameter adaptive algorithm can be developed from the intuitively appealing MIT-Rule, or any similar procedure. Having established the adjustment procedure, stability of the complete adaptive control system can be analyzed using the method of averaging. This will provide conditions for instability which need to be either avoided, or if not, then provide guidelines for control law

modifications.

Taking this approach is not as straight forward as it might sound. There are some pitfalls to avoid and definitely areas where basic research is required. These include the following:

- The theoretical justification for the method of averaging is that a certain design parameter (not one of the adaptive parameters) is sufficiently small. In adaptive systems, this design parameter depends on the speed of adaptation and the magnitude of exogenous inputs such as reference commands and disturbances. Current methods to predict the maximum size of this parameter have been shown by simulation to be very conservative. That is, the stability properties of the adaptive system as obtained from an averaging analysis, are consistent with simulations even for parameter values which greatly exceed the theoretical limit. This is an area which needs to be examined, since we do not want to be overly conservative.

- Averaging analysis works for slow parameter variation, because the parameters are then not being radically affected by any transients. The restrictiveness of slow adaptation can be removed, but this will require a better understanding of the transient behavior of the adaptive system. Some preliminary results utilizing fixed-point theory and Floquet theory have been examined in Kosut and Bitmead(1986, 1987). This is a most difficult problem, but the payoff for LSS systems would be significant.

- Although the method of averaging in principal, involves straightforward calculations, even simple examples can just barely be worked out by hand. It is clear that the level of complexity of a realistic adaptive or otherwise nonlinear system is well beyond hand calculation. Hence, for such powerful analytic methods to be of practical benefit, it is imperative to develop "user-friendly" software tools which provide the requisite nonlinear analysis. At the present time, aside from simulation capability, there are no available software tools for dealing principally with adaptive systems, and certainly none for more general nonlinear systems. This is a research issue in both mathematics and computation, and it is one that may prove essential to some of the tasks we propose to undertake, although it is not a major thrust in our research program. Nonetheless, it is sometimes an overlooked research issue, and thus warrants more than a passing remark.

- The method of averaging so far has been extended and applied only for adaptive *linear* control systems, that is, if the adaptive parameters were held fixed, then the resulting closed-loop system would be linear. This is certainly not the case during rapid slewing maneuvers which involve nonlinear kinematics as well as nonlinear control. Moreover, the control system itself could be nonlinear, despite the fact that certain parameters can be adapted. For example, a time-optimal control of a double integrator (e.g., a single point mass system) is a nonlinear function of position and velocity. Although the method of averaging can handle nonlinear systems in theory, see, e.g., Hale(1969), the application to adaptive *nonlinear* control remains an open area for basic research.

Chapter 3

Identification for Control Design

3.1 Introduction

In order to evaluate a particular model parametrization in system identification, when the intended use of the model is for control design, the analyst needs to specify:

- a representative "true" system
- a model parametrization
- an estimation algorithm
- a set of closed-loop performance criteria
- a contol design approach or rule

3.2 "True" System vs. Model Set

The relation among these choices can be seen by considering the following example. Suppose the "true" system is an undamped elastic rod of length ℓ with a torque $u(t)$ applied at one end. Assuming no external disturbances, the motion of the rod for small angular deflections is well approximated by the partial differential equation

$$m(x)z_{tt}(t,x) + [\rho(x)z_x(t,x)]_x = 0 \tag{3.1}$$

with boundary conditions

$$z_x(t,0) = -u(t)/\rho(0)\,, \quad z_x(t,\ell) = 0 \tag{3.2}$$

where $m(x)$ represents the tortional mass density, $\rho(x)$ represents the tortional stiffness, and $z(t,x)$ is the angular position of the rod at time t and position $x \in [0,\ell]$. Let $y(t)$ denote the output of a sensor which perfectly accurately measures the angular velocity of the rod at the actuator location $x = 0$. Thus,

$$y(t) = z_t(t,0) \tag{3.3}$$

It is clear that the relation between the actuator input $u(t)$ and the velocity sensor output $y(t)$ is given by

$$y(t) = P_o(s)u(t) \tag{3.4}$$

where $P_o(s)$ is the transfer function[1] from $u(t)$ to $y(t)$.

Suppose we now choose as a parametric model of (3.1) a *uniform* undamped elastic rod of length ℓ, with the same torque $u(t)$ applied at one end. Assuming no external disturbances, the motion of the rod for small angular deflections is well approximated by the wave equation [see, e.g., Pichet(1985)]

$$\mu r^2 z_{tt}(t,x) - GJ z_{xx}(t,x) = 0 \tag{3.5}$$

with boundary conditions

$$z_x(t,0) = -\frac{1}{GJ}u(t), \;\; z_x(t,\ell) = 0 \tag{3.6}$$

where μ is mass density, r^2 is radius of gyration, and GJ is the tortional stiffness. Define the *physical parameter* vector

$$\theta = (\alpha, \beta)^T = (\sqrt{\mu r^2 GJ}, \sqrt{\frac{\mu r^2}{GJ}})^T \tag{3.7}$$

Then, the *parametric model* transfer function from $u(t)$ to $y(t)$ is

$$P(s,\theta) = \frac{1}{\alpha}\coth(\beta s) = \frac{1}{\alpha}\frac{1+e^{-2\beta s}}{1-e^{-2\beta s}} \tag{3.8}$$

which can be viewed as a *physical parametrization.*

Unless $m(x)$ and $\rho(x)$ in (3.1) are constant, it is unlikely that there exist any values of α and β in (3.8) such that $P(s,\theta) = P_o(s)$. However, there certainly could be model parameter values which, for some class of functions $m(x)$ and $\rho(x)$, the model error $P(s,\theta) - P_o(s)$ is sufficiently small, in some sense, to satisfy closed loop performance demands. To be more precise about how to measure model error, we briefly state some known results about robustness of feedback systems to model error.

3.3 Conditions for Closed-Loop Stability Robustness

Suppose we have selected a control design rule which for every value of the physical parameter vector θ produces a feedback controller with transfer function $C(s,\theta)$, such that the feedback system consisting of $P(s,\theta)$ and $C(s,\theta)$ is stable and also satisfies the performance demands. Then, this same controller will also stabilize the true system $P_o(s)$ if

$$|P(j\omega,\theta) - P_o(j\omega)| \cdot \left|\frac{C(j\omega,\theta)}{1 + P(j\omega,\theta)C(j\omega,\theta)}\right| < 1 \;, \; \forall \omega \in \mathbf{R} \tag{3.9}$$

This is a typical result arising from robust control theory, see, e.g., *IEEE Transactions on Automatic Control: Special Issue on Robust Control* (1981). More recent results are more extensive involving similar ideas but the formulations would take us too far afield in this discussion, see, e.g.,

[1]The term is used loosely. More precisely, $y(t) = (P_o u)(t)$ where P_o is a convolution operator with kernal $p_o(t)$ whose Laplace transform, or "transfer function", is $P_o(s)$.

Vidyasagar(1985). We remark that a similar result holds for discrete-time feedback systems. In this case, the above condition becomes

$$|P(e^{j\omega},\theta) - P_o(e^{j\omega})| \cdot \left| \frac{C(e^{j\omega},\theta)}{1 + P(e^{j\omega},\theta)C(e^{j\omega},\theta)} \right| < 1 \ , \ \forall \omega \in [-\pi,\pi] \tag{3.10}$$

where we use "z-transforms" rather than Laplace transforms for the discrete-time linear operators P, P_o, and C. We will return to the sampled-data representation in the sequel.

If for feedback design the above inequality is essentially what is needed to insure stability robustness, then there is nothing particularly precious about the physical parametrization of (3.8). One could just as well choose the *canonical* transfer function representation

$$P(s,\theta) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + + a_1 s^{n-1} + \cdots + a_n} \tag{3.11}$$

where now θ is the *physical* parameter vector

$$\theta = (a_1, \ldots, a_n, b_1, \ldots, b_n)^T \tag{3.12}$$

Observe that the physical parameters (α, β) uniquely determine the canonical parameters $(a_1, \ldots, a_n, b_1, \ldots, b_n)$, but the relationship is obviously complicated. The primary disadvantage of using the canonical parameters is that there could be many more than the physical parameters, especially in the case where good model accuracy is required over a wide band of frequencies. As we have mentioned, however, *it is in general much easier to construct numerically stable parameter* estimation algorithms for canonical parameters than for physical parameters.

The above simple example illustrates the difficulties involved in choosing a model parametrization. Obviously the extention to the LSS environment is not so straightforward because of the additional complexity generated by either the large physical size of the structure or the large number of modes within the controller bandwidth. Before discussing these issues we first review some of the known results in the estimation of transfer functions from measured data.

3.4 Review of Transfer Function Estimation

The estimation of a system's transfer function from input-output data has, of course, a long history, and we will not attempt to document that here. There are many excellent survey articles and textbooks that can be referenced, e.g., Jenkins and Watts(1968), Astrom and Eykhoff(1970), *Automatica: Special Issue on Identification*(Jan. 1981), Ljung and Soderstrom(1983), and Ljung(1987), to name a few. These references clearly explain the theory and practice of both parametric and non-parametric methods of transfer function estimation. Parametric methods usually involve the minimization of some time-domain function of the parameters using an iterative or recursive algorithm. Non-parametric methods involve the computation of correlation functions and/or their respective spectral densities. In either case, it is possible to obtain theoretical results which provide asymptotic expressions, as the data record length increases, for the mean-square-error (MSE) of the transfer function estimate, which is precisely the quantity that reflects model accuracy.

3.4.1 Sampled-data Representation and Data Structure

Suppose that the true plant system to be estimated can be described by the discrete-time relation[2]

$$y(t) = P_o(q)u(t) + d(t) \tag{3.13}$$

where $y(t)$ and $u(t)$ are the measured input and output, $P_o(q)$ is the transfer function, and $d(t)$ is the disturbance. It is further assumed that $d(t)$ is the output of a linear system with transfer function $W_o(q)$ which is driven by white noise $\nu_o(t)$ of intensity λ_o. Thus,

$$d(t) = W_o(q)\nu_o(t) \tag{3.14}$$

and hence, has the spectral density

$$S_{dd}(\omega) = |W_o(e^{j\omega})|^2\lambda_o \tag{3.15}$$

The estimation problem is to select out of a model set, an estimate of $P_o(q)$ and $W_o(q)$ from the observed *finite* data record

$$z^N = \{y(t), u(t) : t = 1, ..., N\} \tag{3.16}$$

Observe that if $P_o(s)$ is the true continuous time transfer function, see, e.g., (3.1), then the sampled-data representation is

$$P_o(q) = (1 - q^{-1})\mathcal{Z}\{\frac{1}{s}P_o(s)\} \tag{3.17}$$

where $\mathcal{Z}\{\cdot\}$ denotes the usual z-transform operator.

3.4.2 Prediction Error Methods of Parameter Estimation

Parametric methods of identification proceed by first selecting a set of candidate parametric models of the form:

$$y(t) = P(q,\theta)u(t) + W(q,\theta)\nu(t) \tag{3.18}$$

where $\nu(t)$ is a zero-mean white noise sequence of intensity λ, and $\theta \in \mathbf{R}^p$ is a vector of model parameters which can be either physical or canonical as previously discussed. Observe that if $P(s,\theta)$ is the continuous time transfer function, e.g., (3.8), then the sampled-data representation is

$$P(q,\theta) = (1 - q^{-1})\mathcal{Z}\{\frac{1}{s}P(s,\theta)\} \tag{3.19}$$

Practically every parameter estimation scheme is based on developing an on-line or off-line procedure for selecting the model parameters θ so as to minimize some function of the *prediction error*, [3]

$$\varepsilon(t,\theta) = W^{-1}(q,\theta)[y(t) - P(q,\theta)u(t)] \tag{3.20}$$

[2]We use t to denote sample times, i.e., $t = 0, 1, 2, ...$, etc; q is the shift operator where $qx(t) = x(t+1)$ and $q^{-1}x(t) = x(t-1)$. Also, we use the term transfer function to denote an operator which depends on q.

[3]In this form the prediction error is optimal given the noise assumptions, i.e., the prediction error is the "innovations" sequence corresponding to the steady state Kalman filter tuned to the model and noise statistics. If nothing is known about $\nu(t)$, i.e., $\nu(t)$ is bounded but otherwise unpredictable, then the predictor has the property that as $t \to \infty, \varepsilon(t,\theta) \to \nu(t)$.

3.4.3 Least-Squares Criterion

For example, the *least-squares* estimate is

$$\hat{\theta}_N = \arg\min_{\theta \in \mathcal{F}} V_N(\theta) \tag{3.21}$$

with

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}[L(q)\varepsilon(t,\theta)]^2 \tag{3.22}$$

where $\mathcal{F}$ is a subset of $\mathbf{R}^p$ which consists of those constant θ which stabilize the predictor, and $L(q)$ is a stable filter.

Techniques for computing $\hat{\theta}_N$ involve either iterative or recursive algorithms, see, e.g., Ljung and Sodestrom(1983), Ljung(1987). No matter how it is computed, it can be shown that for large N,

$$\hat{\theta}_N \approx \arg\min_{\theta \in \mathcal{F}} \int_{-\pi}^{\pi} Q_N(\omega,\theta)|P(e^{j\omega},\theta) - \frac{Y_N(\omega)}{U_N(\omega)}|^2 \, d\omega \tag{3.23}$$

with

$$Q_N(\omega,\theta) = |\frac{L(e^{j\omega})U_N(\omega)}{W(e^{j\omega},\theta)}|^2 \tag{3.24}$$

where $Y_N(\omega)$ and $U_N(\omega)$ are the discrete Fourier transforms of $u(t)$ and $y(t)$, respectively, for $t = 1,\ldots,N$. Thus,

$$Y_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} y(t)e^{-j\omega t} \tag{3.25}$$

and the same for $U_N(\omega)$. The above frequency domain expressions support the interpretation that $P(e^{j\omega},\theta)$ is the best quadratic fit of the *empirical transfer function estimate* $Y_N(\omega)/U_N(\omega)$, under the weighting function $Q_N(\omega,\theta)$. Clearly this weight can be effected by the input spectrum $U_N(\omega)$, the data filter $L(e^{j\omega})$, and the noise model $W(e^{j\omega},\theta)$. These choices can be used to design the estimation experiment so as to achieve the smallest possible model error as needed by control design.

3.4.4 Least-Peak Criterion

The estimation criterion can also be taken as

$$V_N(\theta) = \sup_{t \in 1,\ldots,N} |L(q)\varepsilon(t,\theta)| \tag{3.26}$$

Estimators based on this criterion are known to be much sharper when the error is small, i.e., they converge more rapidly than the least-squares estimate. Unfortunately, computational techniques for computing this estimate have not been fully developed.

3.4.5 Computing Model Error: The Mean-Square-Error (MSE)

Suppose we have computed $\hat{\theta}_N$. The next step is to form the transfer function estimates $P(q,\hat{\theta}_N)$ and $W(q,\hat{\theta}_N)$. It is important when using these estimates in control design to be confident of their accuracy as a function of N, the data record length. For example, if $\lim_{N\to\infty}\hat{\theta}_N = \theta_*$, then

we would like to know the convergence rate as well as the properties of the limiting estimates $P(q,\theta_*), W(q,\theta_*)$ with respect to the true system $P_o(q), W_o(q)$. A knowlege of the mean-square-error (MSE) between the estimated transfer function and the true system transfer function would determine the requisite robustness of control that is required to use the transfer function estimate for control design.

Expressions for the MSE are in general quite complicated. It is shown in Ljung(1987), under fairly weak conditions on the model set, that with probability one,

$$\lim_{N\to\infty} \hat{\theta}_N = \theta_* \tag{3.27}$$

where

$$\theta_* = \arg\min_{\theta\in\mathcal{F}} \bar{V}(\theta) \tag{3.28}$$

$$\bar{V}(\theta) = \mathcal{E}[\epsilon^2(t,\theta)] \tag{3.29}$$

Moreover, the quantity

$$\sqrt{N}(\hat{\theta}_N - \theta_*) \tag{3.30}$$

is asymptotically normal with zero mean and covariance matrix

$$R_* = [\bar{V}''(\theta_*)]^{-1} \lim_{N\to\infty} \mathcal{E}\{N[V_N'(\theta_*)]^T V_N'(\theta_*)\}[\bar{V}''(\theta_*)]^{-1} \tag{3.31}$$

where $'$ and $''$ denote differentiation with respect to θ, once and twice, respectively. Thus, for large N, we have

$$\mathcal{E}\{(\hat{\theta}_N - \theta_*)(\hat{\theta}_N - \theta_*)^T\} \approx \frac{1}{N} R_* \tag{3.32}$$

Hence, the MSE for model error is, for large N:

$$\mathcal{E}\{|P(e^{j\omega},\hat{\theta}_N) - P_o(e^{j\omega})|^2\} \approx |P(e^{j\omega},\theta_*) - P_o(e^{j\omega})|^2 + \frac{1}{N} P'(e^{j\omega},\theta_*) R_* P'(e^{j\omega},\theta_*)^T \tag{3.33}$$

This is a very important result, because it provides an analysis tool for computing the estimated transfer function accuracy directly in terms of the transfer function *for any parametrization*. Hence, any proposed estimator whose purpose is to asymptotically produce an estimate which minimizes $\bar{V}(\theta)$, can be evaluated independently from *how* the estimate is produced. In fact, the term above which decays as $1/N$ is the best that can be achieved by any on-line or off-line algorithm.

3.4.6 Non-Parametric Methods of Transfer Function Estimation

Parametric methods as we have discussed them, involve transfer function models which depend on either physical or canonical parameters in a specified way. Methods which do not require such structural knowledge or assumptions are referred to as non-parametric. For example, suppose that the only assumption is that the system has a transfer function which is smooth in some sense.

Available methods for non-parametric transfer function estimation rely on estimating spectral densities. The methods are based on time-series and Fourier analysis of finite data sequences. There are many excellent textbooks on the subject, e.g., Jenkins and Watts(1968), Brillinger(1975), and Priestly (1981).

Just as in the parametric case, it is also possible to obtain expressions for the MSE for model error. A typical approximation for large N is

$$\mathcal{E}\{|\hat{P}(e^{j\omega}) - P_o(e^{j\omega})|^2\} \approx M^2(\gamma)|R(\omega)|^2 + \frac{1}{N}L(\gamma)S_{dd}(\omega)/S_{uu}(\omega) \tag{3.34}$$

where $\hat{P}(e^{j\omega})$ is the estimate of $P_o(e^{j\omega})$ obtained using spectral estimation techniques, and where

$$R(\omega) = P_o''(e^{j\omega}) + P_o'(e^{j\omega})S_{uu}'(\omega)/S_{uu}(\omega) \tag{3.35}$$

with $'$ and $''$ here denoting differentiation with respect to ω, once and twice, respectively. Also

$$M(\gamma) = \int_{-\pi}^{\pi} \omega^2 W_\gamma(\omega)\, d\omega \tag{3.36}$$

$$L(\gamma) = \int_{-\pi}^{\pi} W_\gamma^2(\omega)\, d\omega \tag{3.37}$$

where $W_\gamma(\omega)$ is the *lag window* of width $1/\gamma$. The window is used to generate "smooth" spectral estimates and as seen in the above expressions can be used to adjust the MSE. Typically, as γ increases, the window becomes more narrow, $M(\gamma)$ decreases, and $L(\gamma)$ increases. Thus, as γ increases, the first term (the bias) decreases, but the second term (the variance) increases. Clearly for large N there is an optimal choice of lag window width to minimize the MSE for fixed N, and this can be calculated; see Ljung (1987).

3.5 Application to LSS

The number of actual physical parameters in an LSS is extremely large. Direct parametrizations involving these parameters is out of the question from a practical view. For example, consider the following two modeling approaches: (1) finite element modeling, and (2) continuum modeling.

3.5.1 Finite Element Modeling

A typical finite element model has the form

$$M(\theta)\ddot{q}(t) + K(\theta)q(t) = Bu(t)\,, \quad y(t) = Cq(t) \tag{3.38}$$

where $q(t) \in \mathbf{R}^\ell$ is a vector of nodal displacements (ℓ is large), and $y(t)$ and $u(t)$ are the output and input vectors, respectively. The $\ell \times \ell$ matrices $M(\theta)$ and $K(\theta)$ are the mass and stiffness matrices, respectively. The matrices B and C depend only on the location of the point actuators and point sensors and do not depend on the parameter vector $\theta \in \mathbf{R}^p$. The elements of θ consist of various lengths, areas, mass densities, stiffness values, etc. If θ_0 is a nominal value, then for small parameter deviations, the mass and stiffness matrices can be approximated by [see, e.g., McIntosh(1986)]

$$M(\theta) \approx M_0 + \sum_{i=1}^{p} (\theta_i - \theta_{i0})^{m_i} M_i \tag{3.39}$$

$$K(\theta) \approx K_0 + \sum_{i=1}^{p} (\theta_i - \theta_{i0})^{k_i} K_i \tag{3.40}$$

Suppose that the matrices $\{M_i, K_i : i = 0, \ldots, p\}$ and integers $\{m_i, k_i : i = 1, \ldots, p\}$ are known. Then, the *physical parameters* can be taken as the parameter deviations from the nominal, i.e., $\{\theta_i - \theta_{i0} : i = 0, \ldots, p\}$

3.5.2 Continuum Modeling

It is often possible to approximate a complicated structure with a simpler structure. For example, consider the parametric continuum model

$$m(x,\theta)z_{tt}(t,x) + [\rho(x,\theta)z_{xx}(t,x)]_{xx} = f(t,x) \tag{3.41}$$

where here we take $m(x,\theta)$ as the mass density, $\rho(x,\theta)$ as the elastic modulus, and $f(t,x)$ as the force distribution on the structure. Observe that the parameter vector θ is used to adjust the spatially varying mass density and elastic modulus. In our simple tortional rod example we have assumed uniformity, and thus, the parametrization is given by $m(x,\theta) = \theta_1$, $\rho(x,\theta_2) = \theta_2$, with the parameter vector $\theta^T = (\theta_1, \theta_2)$. More sophisticated functions will undoubtedly be necessary to model LSS. For example, cubic spline functions are examined in Banks and Crowley(1981,1982).

3.5.3 Spatial Measurements

Consider again the wave equation (3.5) with boundary conditions (3.6). If measurements are available of $z_{tt}(t,x)$ and $z_{xx}(t,x)$ at a point $x = x_o$ on the structure, then a natural choice for the prediction error is

$$\varepsilon(t,\theta) = \theta_1 z_{tt}(t,x_o) - \theta_2 z_{xx}(t,x_o) \tag{3.42}$$

Clearly then, estimation of the physical parameters is direct. The problem is that the assumed measurements are not available. These could, however, be estimated from position and rate measurements along the structure. Suppose we have measurements of $z(t,x_o), z(t,x_o \pm \Delta), z_t(t,x_o)$, where Δ is a small spatial distance. Then, an estimate of $z_{tt}(t,x_o)$ is

$$\hat{z}_{tt}(t,x_o) = \left(\frac{s}{s\tau+1}\right) z_t(t,x_o) \tag{3.43}$$

and an estimate of $z_{xx}(t,x_o)$ is

$$\hat{z}_{xx}(t,x_o) = \frac{1}{\Delta^2}[z(t,x_o+\Delta) - 2z(t,x_o) + z(t,x_o-\Delta)] \tag{3.44}$$

The prediction error, based on measurements, which can now be used in the estimation algorithm is

$$\varepsilon(t,\theta) = \theta_1 \hat{z}_{tt}(t,x_o) - \theta_2 \hat{z}_{xx}(t,x_o) \tag{3.45}$$

Thus, the use of spatial measurements can simplify the physical parametrization of transfer function models. We remark that the spatial measurements might not be just position and velocity but might involve the outputs of very many strain gauges placed appropriately on the structure.

3.6 Uncertainty Estimation

The basic idea is to postulate a parametric model with two types of parameters, say $\theta^{ph} \in \mathbf{R}^k$ and $\theta^{aux} \in \mathbf{R}^\ell$. The θ^{ph} parameters correspond to, say, physical parameters in the usual sense of model building, but the θ^{aux} parameters are auxiliary and used to account for poorly known aspects of the system dynamics. To illustrate the point further, let the parametric transfer function model $P(q,\theta)$ have the decomposition

$$P(q,\theta) = \tilde{P}(q,\theta^{ph})\tilde{P}(q,\theta^{aux}) \tag{3.46}$$

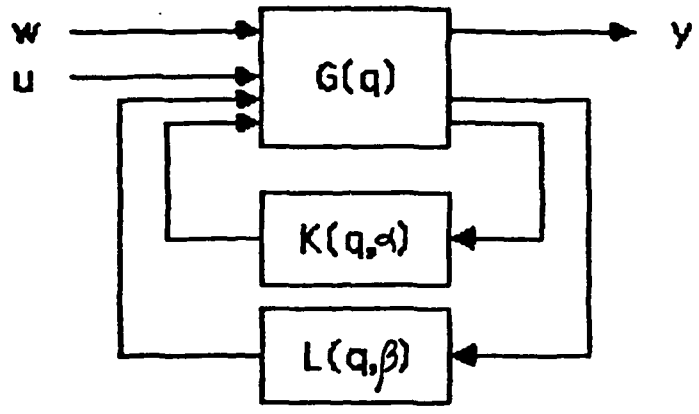

Figure 3.1: A generic parametric model structure

where $\tilde{P}(q,\theta^{ph})$ is the usual parametric model and where $\tilde{P}(q,\theta^{aux})$ is a simple parametric structure which accounts for unmodeled dynamics.

Obviously more elaborate forms can be established depending on the kind of a priori knowledge available regarding the location and type of unmodeled or poorly known dynamics. A proposed generic form is shown in Figure 3.1, where $G(q)$ is a known interconnection transfer matrix, $K(q,\theta^{ph})$ contains all the known parametric stucture, and $L(q,\theta^{aux})$ represents the poorly known dynamics. The true plant would be represented by the corresponding triple $G_o(q), K_o(q), L_o(q)$.

This model is useful for control design if the parameters θ^{ph} and θ^{aux} can be separately calculated as in Kosut(1987a,b). Then a non-parametric method can be used to estimate the model error between the estimated system $G(q), K(q,\hat{\theta}^{ph}), L(q,\hat{\theta}^{aux})$ and the true system $G(q), K_o(q), L_o(q)$. Moreover, the variance of parameter estimates can be utilized in assessing the structure of parameter uncertainty. When this procedure is coupled together with the previously discussed controller sysnthesis methods , we then have an on-line robust control design scheme, what is referred to as *adaptive calibration.*

3.7 Computing the Estimate

In general, adaptive algorithms share a similar genesis and purpose, namely to adjust a parameter vector $\hat{\theta}(t)$ to asymptotically approach a member of the set

$$\mathcal{F}_{opt} = \{\theta \in \mathbf{R}^p : \theta = \arg\min_{\theta} J(\theta)\} \tag{3.47}$$

where

$$J(\theta) = \text{avg}\{\varepsilon^2(t,\theta)\} \tag{3.48}$$

with the averaging operator defined by

$$\text{avg}\{X(t)\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,dt \tag{3.49}$$

The adaptation error $\varepsilon(t,\theta)$ is formed with $\hat{\theta}(t)$ held constant at θ. This error signal can be the prediction error as described before in (3.20), or can be obtained from the adaptive feedback system in Figure 2.3 as the tracking error

$$\varepsilon(t,\theta) = y(t)|_{\hat{\theta}(t)=\theta} - r(t) \tag{3.50}$$

In the ideal case, the parametrization can be selected so as to achieve what is called *perfect matching*, i.e., for some restricted class of exogenous inputs (usually no disturbances), and a sufficient number of parameters (no unmodeled dynamics), the set $\mathcal{F}_{opt}$ has a single member such that $\lim_{t\to\infty} \varepsilon(t,\theta) \to 0$ (or white noise in the stochastic case). This is usually never the case.

Under more realistic conditions, the best that can happen [see, e.g., Anderson et al.(1986)] is that $\hat{\theta}(t)$ asymptotically approaches a small neighborhood of a local minima of $J(\theta)$, i.e., the set

$$\mathcal{F}_* = \{\theta \in \mathbf{R}^p : \frac{\partial}{\partial\theta}J(\theta) = 0,\ \frac{\partial^2}{\partial\theta^2}J(\theta) > 0\} \tag{3.51}$$

3.7.1 Gradient Algorithms

For the purposes of illustration we will assume that parameter adjustments are continuous. In practice the adjustments are at discrete times. Then, assuming continuous adjustments, one method to achieve $\hat{\theta}(t) \to \mathcal{F}_{opt}$ is to adjust $\hat{\theta}(t)$ in the negative gradient direction of the instantaneous squared error, that is, let

$$\dot{\hat{\theta}}(t) \sim -\{\frac{\partial}{\partial\theta}\varepsilon^2(t,\theta)\}|_{\theta=\hat{\theta}(t)} \tag{3.52}$$

Hence, the parameter adaptive algorithm is

$$\dot{\hat{\theta}}(t) = \gamma\psi(t)\varepsilon(t) \tag{3.53}$$

where γ is a positive constant, and

$$\varepsilon(t) = \varepsilon(t,\theta)|_{\theta=\hat{\theta}(t)} \tag{3.54}$$

$$\psi(t) = \psi(t,\theta)|_{\theta=\hat{\theta}(t)} \tag{3.55}$$

with the error gradient

$$\psi(t,\theta) = -\frac{\partial}{\partial\theta}\varepsilon(t,\theta) \tag{3.56}$$

3.7.2 Gauss-Newton Algorithm

If the parameter vector is adjusted to account for second order effects, then the resulting algorithm is

$$\dot{\hat{\theta}}(t) = R^{-1}(t)\psi(t)\varepsilon(t) \tag{3.57}$$

where

$$\dot{R}(t) = \psi(t)\psi^T(t) \tag{3.58}$$

This algorithm has the same form as the recursive-least-squares (RLS) algorithm. The matrix $R(t)$ is approximately the inverse of the covariance of the parameter error.

3.7.3 Constructing the Error Gradient from Measured Signals

A difficulty with constructing the above gradient algorithm is that the error gradient $\psi(t,\theta)$ is a function of the true, but unknown plant. The actual algorithm uses an approximation $\phi(t,\theta)$, and thus

$$\dot{\hat{\theta}}(t) = \gamma\phi(t)\varepsilon(t) \tag{3.59}$$

where $\phi(t)$, referred to as the *regressor*, is given by

$$\phi(t) = \phi(t,\theta)|_{\theta=\hat{\theta}(t)} \tag{3.60}$$

For example, consider the wave model of the uniform rod in (3.8), which can also be expressed as

$$\alpha[y(t) - y(t-2\beta)] = u(t) + u(t-2\beta) \tag{3.61}$$

The parameter vector is $\theta^T = (\alpha\ ,\ \beta)$. Let the adaptive error signal be the *equation error*

$$\varepsilon(t,\theta) = \alpha[y(t) - y(t-2\beta)] - u(t) - u(t-2\beta) \tag{3.62}$$

The negative gradient of the error is then

$$\psi(t,\theta) = \begin{pmatrix} -\varepsilon_\alpha(t,\theta) \\ -\varepsilon_\beta(t,\theta) \end{pmatrix} = \begin{pmatrix} y(t-2\beta) - y(t) \\ -2[\alpha\dot{y}(t-2\beta) + \dot{u}(t-2\beta)] \end{pmatrix} \tag{3.63}$$

If $\dot{y}(t)$ and $\dot{u}(t)$ are not available, then the actual gradient $\psi(t,\theta)$ can be approximated by

$$\phi(t,\theta) = \begin{pmatrix} -\varepsilon_\alpha(t,\theta) \\ -\varepsilon_\beta(t,\theta) \end{pmatrix} = \begin{pmatrix} y(t-2\beta) - y(t) \\ -\left(\frac{2s}{s\tau+1}\right)[\alpha y(t-2\beta) + u(t-2\beta)] \end{pmatrix} \tag{3.64}$$

where the filter time constant τ is a design choice.

3.7.4 Algorithm Modifications

To insure that adaptive algorithms are working, one often includes various "safety nets" or AI features. For example, in uncertainty estimation, the controller is not changed until some level of confidence is established. Other AI features include monitoring of information, covariance monitoring, perisitent excitation conditioning, and parameter projection, to name a few. Various normalizations of the measured sugnals also enhance the ability of adaptive systems to insure bounded parameter values. These latter approaches need prior information regarding model error and parameter ranges.

As an example, consider the Gauss-Newton algorithm with "covariance" resetting. That is, set

$$R(t_i) = R_o = R_o^T > 0 \tag{3.65}$$

where the reset times $\{t_i\}$ are found from

$$t_i = t_{i-1} + \min\{\delta : \int_{t_{i-1}}^{t_{i-1}+\delta} \psi(t)\psi^T(t)\,dt \geq \alpha\} \tag{3.66}$$

where $t_0 = 0$. These reset times are essentially those where sufficient information is gathered. Usually the controller is adjusted only at these times, i.e., controller adjustment takes place when the model estimate is good.

3.7.5 Performance of the Algorithm

Despite the above approximation, from a practical point of view it is certainly acceptable that the parameters approach and remain in a small neighborhood of $\mathcal{F}_*$, provided that members of this set also produce acceptable performance.

Assuming this is so, let $\theta_* \in \mathbf{R}^p$ denote such a setting, of which there could be many. We refer to each θ_* as a *tuned parameter* and to the corresponding feedback system as the *tuned system* [see, e.g., Kosut and Friedlander(1985)]. Clearly the tuned system is the feedback system above with $\hat{\theta}(t)$ fixed at θ_*. We can now pose the following questions regarding the adaptive system :

1. How do the tuned parameters depend on the exogenous inputs?
2. Is the adaptive system stable in a neighborhood of the tuned system. Furthermore, how small is this neighborhood?
3. What is the region of attraction to a small neighborhood of the tuned system.
4. What is the rate of convergence to this small neighborhood.

Some of these questions can be answered by the method of averaging which is discussed in the following section.

3.8 Averaging Analysis of Stability and Convergence

3.8.1 Method of Averaging

The classical method of averaging applies to a differential equation of the form

$$\dot{x} = \gamma f(t,x)\,,\ \gamma > 0 \tag{3.67}$$

where γ is a positive constant. If γ is sufficiently small then under suitable regularity conditions on $f(t,x)$, the stability of the above time-varying system is inherited from the stability of the simpler autonomous system

$$\dot{x} = \gamma f_a(x) \tag{3.68}$$

where

$$f_a(x) = \text{avg}\{f(t,x)\} \tag{3.69}$$

This system is referred to as the *averaged* system. Observe that for $f_a(x)$ to exist, some other restrictions must also apply to $f(t,x)$, e.g., $f(t,x)$ is almost periodic in t uniformly for x in a compact set; although weaker conditions can be stated.

The above type of result is certainly expected on intuitive grounds, provided that $f(t,x)$ is smooth enough and the average value $f_a(x)$ exists. A precise formulation of the stability-instability conditions is given in Theorem V.3.1, Hale(1969).

In the adaptive system we examine the "average" parameter trajectories ensuing from

$$\dot{\theta}_a = \gamma f(\theta_a) \tag{3.70}$$

where the adaptive parameter vector field is given by

$$f(\theta) = \text{avg}\{\phi(t,\theta)\varepsilon(t,\theta)\} \tag{3.71}$$

The question is then to determine conditions under which the behavior of the actual parameter trajectories is like that of the average parameter trajectories. Using the above averaging theory, analysis of the adaptive system can be divided into answering questions about its asymptotic and transient characteristics.

3.8.2 Asymptotic Analysis

The definition of the tuned system as well as the stability of the adaptive system in the neighborhood of the tuned system can be answered by Theorem 4.2 in Anderson et al.(1986) or Section 3 in Bodson et al.(1985). The flavor of these results can be stated as follows:

THEOREM 1 *Let the tuned parameter set be defined as those $\theta_* \in \mathbf{R}^p$ which stabilize the feedback system and which also satisfy*

$$\text{avg}\{f(t,\theta_*)\} = 0 \tag{3.72}$$

If the functions $f(t,\theta), \phi(t,\theta)$ and $\varepsilon(t,\theta)$ are sufficiently smooth, almost periodic in t uniformly for θ in compact sets, then for a sufficiently small adaptation gain γ, and a sufficiently small peak value of the tuned error signal $\varepsilon(t,\theta_*)$, solutions of the adaptive system originating in a small neighborhood of the tuned solutions $\theta_*, \phi(t,\theta_*)$, and $\varepsilon(t,\theta_*)$, will remain there if

$$\max_i \text{Re}\lambda_i[B(\theta_*)] < 0 \tag{3.73}$$

and, moreover, will leave there if

$$\max_i \text{Re}\lambda_i[B(\theta_*)] > 0 \tag{3.74}$$

where the matrix function $\theta \mapsto B(\theta)$ is given by

$$B(\theta) = \frac{\partial}{\partial\theta}\text{avg}\{f(t,\theta)\} \tag{3.75}$$

In addition, if the tuned solution is stable, that is, if $\max_i \text{Re}\lambda_i[B(\theta_*)] < 0$, then

$$\limsup_{t\to\infty} \|\hat{\theta}(t) - \theta_*\| = O(\limsup_{t\to\infty} |\varepsilon_*(t)|) + O(\gamma) \tag{3.76}$$

$$\limsup_{t\to\infty} \|\eta(t)\| = O(\gamma) \tag{3.77}$$

The type of smoothness conditions required of the functions $f(t,\theta), \phi(t,\theta)$, and $\varepsilon(t,\theta)$ is not severe, e.g., continuity in t and Lipschitz continuity in θ for θ in compact sets.

The sharp stability-instability boundary expressed above allows not only for an assessment of a particular design, but also indicates how to modify and improve the algorithm.

3.8.3 Transient Analysis

An understanding of the transient properties of the adaptive system requires answering the questions posed before, namely, determining the region of attraction to a small neighborhood of the

tuned system, and the rate of convergence. The folowing result, based on averaging, can be found in Anderson et al.(1986).

THEOREM 2 *Under the conditions stated in Theorem 1, if the initial parameter value $\hat{\theta}_o$ is strictly inside a convex subset of the constant parameter stability set, then $\hat{\theta}(t)$ will remain in that set and converge exponentially at a rate no slower than $O(\gamma)$ to the small neighborhood of the tuned system defined in Theorem 1.*

Some of the restrictions can be relaxed, see, e.g., Riedle and Kokotovic(1986) or Bodson et al.(1985). Specifically, the region of attraction can be relaxed to a compact subset of the constant parameter stability set; also, estimates of the rate of convergence far from the tuned setting do not have to be $O(\gamma)$.

These type of results are pleasing from an intuitive point of view, thus providing a *qualitative* analysis. However, they do not apply when the parameters leave the constant parameter stability set, nor do they provide *quantitative* performance measures. In addition, we still require slow adaptation, which in itself is not too objectionable, perhaps only cautious, but again specific bounds on the adaption gain extracted from the theory tend to be extremely conservative.

3.9 References

"Active Control of Space Structures (ACOSS)", Phase 1a final report, prepared by LMSC for Darpa under contract F30602-80-C-0087, Aug. 1981.

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, (1986),*Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

B.D.O. Anderson and R.M. Johnstone (1983), "Adaptive Systems and Time- Varying Plants", *Int. J. Control*, Vol. 37, No. 2, pp.367-377.

B.D.O. Anderson and J.B. Moore, (1979), *Optimal Filtering*, Prentice-Hall, NJ.

K.J.Astrom (1983),"Analysis of Rohrs Counter-Examples to Adaptive Control", *Proc. 22nd IEEE Conf. on Dec. and Contr.*, San Antonio, TX.

K.J.Astrom (1984),"Interactions Between Excitation and Unmodeled Dynamics in Adaptive Control",*Proc. 23rd IEEE Conf. on Dec. and Contr.*,Las Vegas, NV, pp.1276-1281.

K.J. Astrom and P. Eykhoff (1971), "System Identification—A survey", *Automatica*, 7:123–167.

M. Athans et al. (1977), "The Stochastic Control of the F-8C Aircraft Using A Multiple Model Adaptive Control (MMAC) Method", *IEEE Trans. Aut. Contr.*, vol AC-22, no 5, pp 768-780, Oct.

J.N. Aubrun et al. (1979),"Stability Augmentation for Flexible Structures", *Proc. IEEE CDC*, Hollywood, FL, Dec. 1979.

H.T. Banks, J.M. Crowley, and K. Kunisch, (1983) "Cubic Spline Approximation Techniques for Parameter Estimation in Distributed Systems", *IEEE Trans. on Aut. Contr.*, vol. AC-28, no. 7, July 1983.

W. H. Bennett, H.G. Kwatny, and J.S. Baras, (1987), "Robustness Issues in Boundary Feedback Control of Flexible Structures", *Proc. 1987 ACC*, pp. 1307-1313, Minn., MN, June 1987.

A. Bhaya and C.A. Desoer, (1985),"On the Design of Large Flexible Space Structures (LFSS)", *IEEE Trans. Aut. Contr.*, vol AC-30, no 11, Nov. 1985.

N.N. Bogoliuboff and Y.A. Mitropolskii (1961) *Asymptotic methods in the theory of Nonlinear Oscillators*, Gordon and Breach, New York.

M. Bodson, S. Sastry, B.D.O. Anderson, I. Mareels, and R.R. Bitmead, (1986), "Nonlinear Averaging Theorems, and the Determination of Parameter Convergence Rates in Adaptive Control", *Systems and Contr. Letters*, to appear.

A. Bryson (1977), Guest Editorial in Mini-Issue on NASA's Adavanced Control Law Program for the F-8 DFBW Aircraft, *IEEE Trans. Aut. Control*, Vol. AC-22, No.5, Oct. 1977.

M. Bodson, S. Sastry, B.D.O. Anderson, I. Mareels, and R.R. Bitmead, (1986), "Nonlinear Averaging Theorems, and the Determination of Parameter Convergence Rates in Adaptive Control", *Systems and Contr. Letters*, to appear.

S. Boyd et al. (1986), "A New CAD Method and Associated Architectures for Linear Controllers", ISL Tech. Report L-104-81,1, Stanford University, Dec. 1986.

D.R. Brillinger (1975), *Time Series: Data Analysis and Theory*, Holt, Rinehart, and Wintron, New York.

C.A. Desoer, R.W. Liu, J. Murray and R. Sacks (1980), "Feedback System Design: The Fractional Representation Approach to Analysis and Synthesis", *IEEE Trans. Aut. Control*, Vol. AC-25, pp. 399–412, June 1980.

C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, 1975.

J.C. Doyle and C.C. Chu (1985), "Robust Control of Multivariable and Large Scale Systems", *Honeywell SRC: Final Technical Report* on AFOSR Contract F49620-84-C-0088, March 1986.

E. Denman et al. (1986), *Identification of Large Space Structures on Orbit*, Final Report AFRPL-TR-86-054, F04611-85-C-0092.

B. Egardt (1979), *Stability of Adaptive Systems*, Springer-Verlag.

R.E. Goodson and M. Polis (1974), ed., *Identification of Parameters in Distributed Systems*, The American Society of Mechanical Engineers, New York.

G.C. Goodwin and K.S. Sin (1984), *Adaptive Filtering, Prediction, and Control*, Prentice-Hall, New Jersey.

G.C. Goodwin and R.L. Payne, (1977) *Dynamic System Identification: Experiment Design and Data Analysis*, Academic Press, New York

P. Hagerdorn (1985), "On a New Concept of active Vibration Damping of Elastic Structures", presented at *Structural Dynamics Symp.*, Univ. of Waterloo, July 1985.

J.K. Hale (1980), *Ordinary Differential Equations*, Kreiger, Molaban, FL, originally published (1969), Wiley (Interscience), New York.

A.J. Helmicki, C.A. Jacobson, and C.N. Nett, (1987),"Fundamentals of Practical Controller Design for LTI Plants with Distributed Parameters", *Proc. 1987 ACC*, pp 1203-1208, Minn., MN, June 1987.

P.A. Ioannou and P.V. Kokotovic (1983) *Adaptive Systems with reduced models*, Springer-Verlag, New york.

D.J.G. James (1971),"Stability of a Model Reference Control System", *AIAA Journal*, Vol. 9, pp.950-952.

R.L. Kosut, B.D.O. Anderson, and I.M.Y. Mareels,(1987), "Stability Theory for Adaptive Systems: Methods of Averaging and Persistency of Excitation", *IEEE Trans. on Aut. Contr.*, Vol. AC-32, No. 1, pp. 20–34, Jan. 1987.

R.L.Kosut, I.M.Y.Mareels, B.D.O.Anderson, R.R.Bitmead, and C.R. Johnson, Jr., (1987),"Transient Analysis of Adaptive Control", *Proc. IFAC, 10th World Congress*, Munich, Germany, July 1987.

R.L. Kosut and R.R. Bitmead (1986), "Fixed Point Theorems for the Stability Analysis of Adaptive Systems", *IFAC Workshop on Adaptive Control*, AC-30:834.

R.L. Kosut and B.D.O. Anderson,(1986), "Local Stability Analysis for a Class of Adaptive Systems", *IEEE Trans. Auto. Contr.*, Jan. 1986.

R.L. Kosut and B. Friedlander (1985), "Robust Adaptive Control: Conditions for Global Stability", *IEEE Trans. on Aut. Contr.*, AC-30(7):610–624.

R.L. Kosut and C.R. Johnson, Jr. (1984), "An Input-Output View of Robustness in Adaptive Control", *Automatica*, 20(5)569–581.

R.L. Kosut, (1986), "Adaptive Calibration: An Approach to Uncertainty Modeling and On-Line Robust Control Design", *Proc. 25th IEEE CDC*, Athens, Greece, Dec. 1986.

R.L. Kosut (1987a), "Adaptive Uncertainty Modeling", *Proc. 1987 ACC*, June 10-12, 1987, Minn., MN.

R.L. Kosut (1987b), "Adaptive Control of Large Space Structures: Uncertainty Estimation and Robust Control Calibration", *Large Space Structures: Dynamics and Control*, ed. S.N. Atluri and A.K. Amos, Springer-Verlag, to appear.

R.L. Kosut, H. Salzwedal, and A. Emami, (1983), "Robust Control of Flexible Structures", *AIAA J. Guid. Contr.*, vol 6, no 2, Mar-Apr 1983.

L. Ljung and T. Soderstrom (1983), *Theory and Practice of Recursive Identification*, MIT Press.

L. Ljung (1977),"On Positive real Transfer functions and the Convergence of some Recursive Schemes",*IEEE Trans. Auto. Control*, Vol. AC-22, pp.539- 551.

S.D. McIntosh, Jr. and M.A. Floyd, (1985), "Investigation of Interactive Structural and Controller Synthesis for Large Spacecraft", final report for AFOSR contract F49620-84-C-0025.

I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, M. Bodson, and S. Sastry (1986), "Revisiting the MIT-Rule for Adaptive Control", *Proc. 2nd IFAC Workshop on Adaptive Systems*, Lund, Sweden.

R.V. Monopoli (1974),"Model Reference Adaptive Copntrol with an Augmented Error Signal", *IEEE Trans. Auto. Control*, AC-19, pp.474-484.

K.S. Narendra, Y.H. Lin, and L. Valavani, "Stable Adaptive Control Design, Part II: Proof of Stability", *IEEE Trans. Auto. Control*, Vol. AC-25, No. 3, pp.440-449.

C.N. Nett, C.A. Jacobson, and M.J. Balas, (1984), "A Connection Between State Space and Doubly Coprime Fractional Representations", *IEEE Trans. Aut. Contr.*, vol 29, pp831-832.

P.V. Osburn, H.P. Whitaker, and A. Kezer (1961),"New Developments in the Design of Model Reference Adaptive control", *Inst. Aeronautical Sciences*, Paper 61-39.

P.C. Parks (1966),"Lyapunov Redesign of Model Reference Adaptive Control Systems", *IEEE Trans. Auto. Control*, Vol. AC-11, pp.362-367.

R. Pichet, (1985), *Frequency Domain Continuum Modeling and Control of Third-Generation Spacecraft*, Tech. Report, Univ. of Waterloo, Ontario, Canada.

B.D. Riedle and P.V. Kokotovic (1986), "Integral Manifolds of Slow Adaptation", *IEEE Trans. Aut. Control*, Vol. 31, No. 4, pp. 316–324, April 1986.

B.D. Riedle and P.V. Kokotovic (1985), "A Stability-Instability Boundary for Disturbance-Free Slow Adaptation and Unmodeled Dynamics", *IEEE Trans. on Aut. Contr.*, AC-30:1027–1030.

C.E.Rohrs, L.S.Valavani, M.Athans, and G.Stein (1985),"Robustness of Continuous Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics",*IEEE Trans. Aut. Contr.*, AC-30(9):881-889, Sept.

E. Rafajlowicz (1983), "Optimal Experiment Design for Identification of Linear Distributed-Parameter Systems: Frequency Domain Approach", *IEEE Trans. Aut. Contr.*, vol AC-28, no.7, July 1983.

M.G. Safonov, A.L. Laub, and G.L. Hartmann, (1981),"Feedback Properties of Multivariable Systems: The Role and Use of The Return Difference Matrix", *IEEE Trans. Aut. Contr.*, vol AC-26, Feb. 1981.

D.B. Schaechter (1982), "Estimation of Distributed Parameter Systems", *AIAA J. Guid. and Contr.*, vol 5, no 1, Jan-Feb 1982.

D.B. Schaechter (1986), "Estimation of Distributed Parameter Systems: Some Closed Form Solutions", *AIAA J. Guid. and Contr.*, vol 9, no 4, July-Aug 1986.

A.H. von Flotow, (1984), *Disturbance Propagation in Structural Networks: Control of Large Space Structures*, Ph.D. Dissertation, Stanford University, Stanford, CA, June 1984.

M. Vidyasagar,(1984), "The Graph Metric for Unstable Plants and Robustness Estimates for Feedback Stability", *IEEE Trans. Auto. Contr.*, **AC-29**:403–418, May 1984.

M. Vidyasagar (1985), *Control system Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

H.P. Whitaker (1959),"An Adaptive System for the Control of Aircraft and Spacecraft", *Inst. Aeronautical Sciences*, Paper 59-100.

D.C. Youla, H.A. Jabr, and J.J. Bongiorno, Jr. (1976), "Modern Wiener-Hopf Design of Optimal Controllers, Part II: The Multivariable Case", *IEEE Trans. Aut. Contr.*, vol AC-21, pp 319-338, June 1976.

G. Zames and B.A. Francis, (1983) "A New Approach to Classical Frequency Methods: Feedback and Minimax Sensitivity", *IEEE Trans. Auto. Contr.*, **AC-28**:585–601, May 1983.

G. Zames, "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms and Approximate Inverses", *IEEE Trans. Auto. Contr.*, **AC-26**:301–320, April 1981.

Appendix A

Reprints of Selected Publications

A few recent publications are appended in this section.

Large Space Structures: Dynamics & Control
ed. S.N. Atluri, A.K. Amos, Springer-Verlag (to appear 1987)

Adaptive Control of Large Space Structures:

Uncertainty Estimation and Robust Control Calibration[1]

Robert L. Kosut [2]

Abstract

An approach is presented to the problem of designing a robust control using on-line measurements. The idea is to use standard methods of parametric system identification to obtain a nominal estimate of the plant transfer function. Non-parametric spectral methods are then used to obtain a frequency domian expression for model uncertainty. If the model uncertainty exceeds a specified frequency bound, which has been predetermined from the nominal model and the performance criteria, then data filters used in the system identification are modified and the procedure is repeated. An analysis is presented which establishes conditions under which the procedure will actually converge to a satisfactory robust design. An example is provided which illustrates the method and supporting analysis.

1 Introduction

Large space structure (LSS) systems impose stringent performance demands, and hence, feedback designs will of necessity be based on very accurate models. However, the on-orbit dynamics will not be sufficiently like those obtained from either ground-testing or even from sophisticated computer generated modeling techniques, e.g., finite element modeling. Therefore, it is necessary to be able to identify the LSS dynamics directly from on-orbit measurements, and simultaneously, tune or re-design the control. Hence, the control design cycle is an *adaptive* process, typically starting with a nominal low-performance design based on a coarse model, and then re-designed from on-orbit data.

This paper addresses some of the issues involved in the adaptive control of LSS systems, specifically the problem of vibration suppression.

1.1 Adaptive Control

Adaptive control, as depicted in Figure 1, essentially consists of two processes, namely: (1) a model parameter estimator which uses a finite data record of input-output measurements, and (2) a control design rule which transforms the model parameter estimates into control parameters.

The procedure can go awry mainly because the model estimate is not sufficiently accurate and the control design rule effectively assumes that the estimates are perfectly accurate. Although the procedure may work well whenever the parameter estimates are in a good region, they may never get there.

One route around this problem is to prove that it can never occur, which involves *analyzing* the complete adaptive system. For example, analysis techniques based on the method of averaging can provide an assessment of the adaptive systems robustness, e.g., Anderson et al.(1986).

[1]Research supported by AFOSR, Directorate of Aerospace Sciences, under contract F49620-85-C-0094.

[2]R.L. Kosut is a Senior Scientist with Integrated Systems, Inc., 2500 Mission College Blvd., Santa Clara, CA 95054, and a Consulting Professor with the Information Systems Lab, Stanford U..iversity, Stanford, CA 95305.

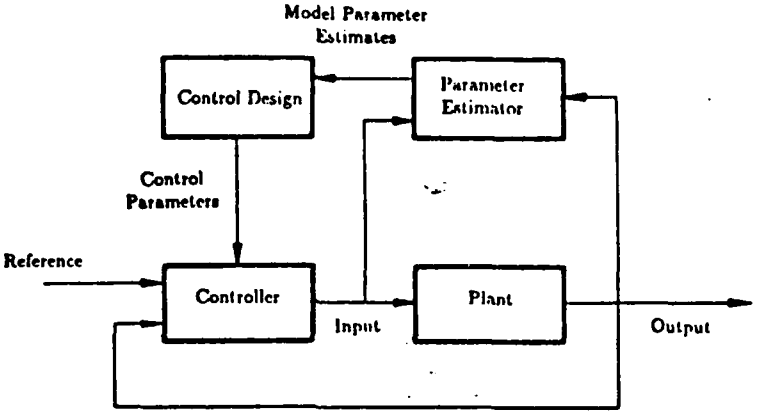

Figure 1: Adaptive Control

Another route, explored here, is to modify the process so that along with estimating model parameters, a measure of model uncertainty is also obtained. With such an estimate in hand the control design rule can be modified to account for the model uncertainty. If the resulting controller is too cautious, because the model estimate is too uncertain, then it is necessary to repeat the estimation procedure so as to obtain a better model.

In this paper we will concentrate on a particular procedure for obtaining a measure of model uncertainty for linear time invariant plants. The process is referred to as *adaptive uncertainty modeling* and the outcome is a frequency domain expression for the model uncertainty. The motivation arises from robust control theory which utilizes frequency domain expressions for characterizing a *set of uncertainty* within which lies the true plant, e.g., Francis and Zames(1984), Safonov et al.(1981), Doyle and Chu(1986), Vidyasagar(1985). An adaptive control system, modified so that it produces a set of uncertainty, is depicted in Figure 2.

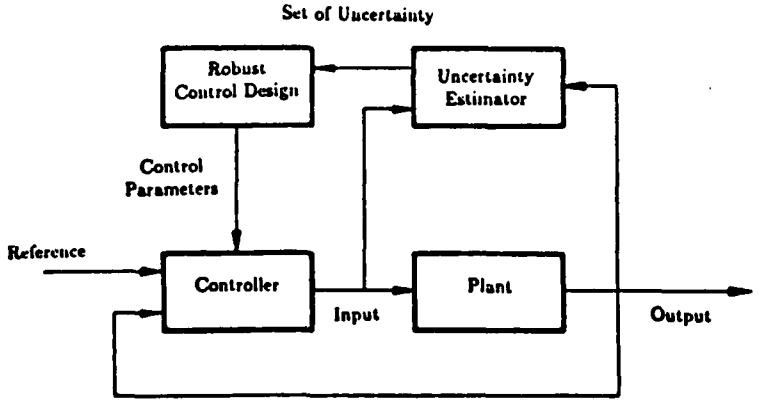

Figure 2: Adaptive Control with Uncertainty Estimator

The indicated adaptive process is referred to here as *adaptive calibration*, Kosut(1986,1987). The set of uncertainty is extracted from input-output measurements. We will describe one approach to this problem which uses a combination of parametric prediction error methods together

with standard non-parametric (spectral) estimation methods. We then show how the set of uncertainty produced by the modified identifier allows for performance evaluation *before* the controller is adjusted. If the predicted performance is not satisfactory then the identification process is repeated under different conditions which will reduce model uncertainty where needed. The resulting controller is designed to be robust with respect to the estimated set of uncertainty. Thus, a large set of uncertainty requires a cautious or low authority controller, whereas a small set of uncertainty will result in a high authority controller. Observe that the order of the controller can vary with the set of uncertainty.

This paper is organized as follows: Section 2 develops the sampled-data linear transfer function model. Section 3 states some known results regarding stabilization and robustness of linear control systems. Section 4 is the main section, decribing in detail an approach to estimating the set of uncertainty and a procedure for adaptive calibration for both open-loop and closed-loop situations. Section 5 presents an example of uncertainty estimation from experimental data. Section 6 contains concluding remarks.

2 Linear Flexible Dynamics

2.1 Transfer Function Model

Under the assumptions of linear elasticity and small deflections, at any time t and position $\mathbf{r}$ on the structure, the deflections are given by

$$y(t,\mathbf{r}) = \sum_{k=0}^{\ell} \eta_k(t)\psi_k(\mathbf{r}) \tag{1}$$

where $\psi_k(\mathbf{r})$ is the kth mode shape at position $\mathbf{r}$, and $\eta_k(t)$ is the kth modal amplitude as a function of time. The upper limit ℓ in the above sum is theoretically infinite, but for all practical purposes can be considered here to be finite, but extremely large. Assuming there are point actuators located at the discrete positions $\mathbf{r}_{a1},\ldots,\mathbf{r}_{am_a}$, then the modal amplitudes each satisfy

$$\ddot{\eta}_k(t) + 2\zeta_k\Omega_k\dot{\eta}_k(t) + \Omega_k^2\eta_k(t) = \sum_{i=1}^{m_a} u_i(t)\psi_k(\mathbf{r}_{ai}),\ k = 1,2,\ldots,\ell \tag{2}$$

where Ω_k and ζ_k are the kth modal frequency and damping, respectively, and where $u_i(t)$ is the ith actuator force or torque on the structure. A position sensor located at $\mathbf{r}$ would measure

$$y_p(t,\mathbf{r}) = \sum_{k=0}^{\ell} \eta_k(t)\psi_k(\mathbf{r}) \tag{3}$$

Likewise, a velocity sensor at $\mathbf{r}$ would measure

$$y_v(t,\mathbf{r}) = \sum_{k=0}^{\ell} \dot{\eta}_k(t)\psi_k(\mathbf{r}) \tag{4}$$

Suppose that actuators are placed at $\mathbf{r}_{a1},\ldots,\mathbf{r}_{am_a}$, position sensors at $\mathbf{r}_{p1},\ldots,\mathbf{r}_{pm_p}$, and velocity sensors at $\mathbf{r}_{v,1},\ldots,\mathbf{r}_{v,m_v}$. Then, the transfer function from the ith applied force to the jth position or velocity sensor output is then, respectively

$$G_{y_{pj}u_i}(s) = \sum_{k=0}^{\ell} \psi_k(\mathbf{r}_{pj})\psi_k(\mathbf{r}_{ai})\frac{1}{s^2 + 2\zeta_k\Omega_k s + \Omega_k^2} \tag{5}$$

(6)

$$G_{y_{vj}u_i}(s) = \sum_{k=0}^{\ell} \psi_k(\mathbf{r}_{vj})\psi_k(\mathbf{r}_{ai})\frac{s}{s^2 + 2\zeta_k\Omega_k s + \Omega_k^2} \tag{7}$$

For purposes of illustration, suppose that there is only one position sensor at $\mathbf{r}_p$ and one actuator, not located at the same position as the sensor, but at $\mathbf{r}_a$. Let $y(t)$ denote the output of the position sensor. Since disturbance forces act on the structure and the sensors have some noise component, the actual sensor output will read [3]

$$y(t) = G(s)u(t) + d(t) \tag{8}$$

where $d(t)$ is the cumulative effect of disturbances and noise sources as seen at the output, and where $G(s)$ is the transfer function from actuator to sensor location. Thus,

$$G(s) = \sum_{k=1}^{\ell} \frac{\beta_k}{s^2 + 2\zeta_k\Omega_k s + \Omega_k^2} \tag{9}$$

where

$$\beta_k = \psi_k(\mathbf{r}_p)\psi_k(\mathbf{r}_a) \tag{10}$$

2.2 Sampled-Data Representation

Assume that control commands and data acqusition occur at discrete sampling instances separated by uniform sampling intervals of duration $t_{sa} = 1/\Omega_{sa}$ seconds, where Ω_{sa} is the sampling frequency in Hz. Thus, $u(t)$ is constant between samples, i.e., for any integer k,

$$u(t) = u(t_k),\ t \in [t_k, t_{k+1}) \tag{11}$$

where $t_k = kt_{sa}$. If, in addition, the sensor output is sampled at the same instances, then the discrete-time representation of (8) is

$$y(t_k) = P_0(q)u(t_k) + d(t_k) \tag{12}$$

where $P_0(q)$ is the *zero-order hold equivalent* of $G(s)$, expressed as a function of the *shift operator* q defined by $qx(t_k) = x(t_{k+1})$ and $q^{-1}x(t_k) = x(t_{k-1})$. Thus, in terms of the usual "z-transform" operator $\mathcal{Z}(\cdot)$, we have

$$P_0(q) = (1 - q^{-1})\mathcal{Z}\{P(s)/s\} \tag{13}$$

Using (9) gives

$$P_0(q) = \sum_{k=1}^{\ell}(1 - q^{-1})\mathcal{Z}\{\frac{1}{s}\frac{\beta_k}{s^2 + 2\zeta_k\Omega_k s + \Omega_k^2}\} \tag{14}$$

In order to simplify the notation in the sequel, time will be normalized with respect to the sampling instants. That is, the discrete-time representation of the LSS will be written compactly as

$$y(t) = P_0(q)u(t) + d(t) \tag{15}$$

where t takes on integer values only, i.e., $t = 1, 2, \ldots$, and the shift operator is normalized so that $qx(t) = x(t+1)$ and $q^{-1}x(t) = x(t-1)$. We will take (15) as the *true* LSS system and the measured data record as

$$z^N = \{y(t), u(t) : t = 1, \ldots, N\} \tag{16}$$

[3] The variable s is used to denote either the Laplace transform variable or the differential operator d/dt, depending on the context.

2.3 Disturbance Spectrum

Assume that $d(t)$ is a zero-mean sequence with *spectral density* $S_{dd}(\omega)$, where ω is normalized frequency in the interval $[-\pi, \pi]$. The spectral density is defined as the Fourier transform of the *auto-correlation* function $R_{dd}(\tau)$, that is

$$S_{dd}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{dd}(\tau) e^{-j\omega\tau} \tag{17}$$

where, following Ljung(1987),

$$R_{dd}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathcal{E}\{d(t)d(t+\tau)\} \tag{18}$$

where $\mathcal{E}\{\cdot\}$ is the usual expectation operator. Observe that if $d(t)$ is either a stationary stochastic process or a periodic deterministic sequence, then the auto-correlation function as defined above will exist. Such sequences are said to be *quasi-stationary*.

A convenient representation of the disturbance is that it is the output of a stable system with transfer function $H_0(q)$ and input $w(t)$, a zero-mean sequence with constant spectrum $S_{ww}(\omega) = \sigma_0^2$, e.g., white noise. Hence, the auto-correlation is an impulse function, and furthermore

$$S_{dd}(\omega) = \sigma_0^2 |H_0(e^{j\omega})|^2 \tag{19}$$

3 Linear Control Design

3.1 Stabilization and Robustness

Before designing an adaptive control we first consider the problem of designing a *robust* non-adaptive control where the following information is assumed to be known: (i) a nominal model of the LSS, and (ii) a description of the accuracy of the model, e.g., model error as a function of frequency. In this section we state some known results regarding the stabilization and robustness properties of feedback systems using the above information.

Consider the linear scalar feedback system depicted in Figure 3, with reference input $r(t)$, disturbance $d(t)$, control actuator input $u(t)$, and sensor output $y(t)$. The transfer functions of the plant and feedback compensator are $P(q)$ and $C(q)$, respectively.

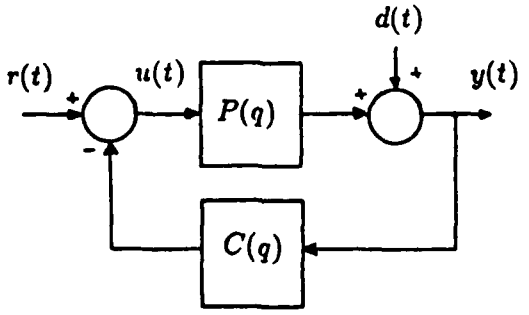

Figure 3: Linear Feedback System

Following Vidyasagar(1985), let S denote the set of transfer functions which are rational in q^{-1} (ratios of polynomials in q^{-1} with real coefficients) and stable (all poles strictly inside the unit

disc). We say that C *stabilizes* P if and only if the three transfer functions $C/(1+PC)$, $1/(1+PC)$, and $P/(1+PC)$ are in $\mathbf{S}$. We now state the following known result.

THEOREM 1

(i) Stabilization

If $P \in \mathbf{S}$ then C stabilizes P if and only if $C = Q(1-PQ)^{-1}$ for some $Q \in \mathbf{S}$.

(ii) Robustness

If

$$P = \hat{P} + \Delta \tag{20}$$

and if some compensator $\hat{C}$ stabilizes $\hat{P}$, then it also stabilizes P if and only if

$$\Delta(1+\Delta\hat{Q})^{-1} \in \mathbf{S} \tag{21}$$

where $\hat{Q} = \hat{C}(1+\hat{P}\hat{C})^{-1} \in \mathbf{S}$.

The first part of the theorem essentially asserts that if P is stable then the set of *all* rational compensators that stabilize P, denoted by $\mathcal{S}(P)$, is given by

$$\mathcal{S}(P) = \{Q(1-PQ)^{-1} : Q \in \mathbf{S}\} \tag{22}$$

Thus, *any* stabilizing controller must produce a Q which is in $\mathbf{S}$.

The second part of the theorem provides conditions under which the feedback system consisting of the nominal plant $\hat{P}$ and the stabilizing control $\hat{C}$, which can be thought of as the *nominal* closed-loop system, is robustly stable to a dynamical plant perturbation Δ. Observe that in this part of the theorem, both the nominal plant model $\hat{P}$ and the perturbation Δ, may be unstable. However, when $\Delta \in \mathbf{S}$, condition (21) reduces to

$$(1+\Delta\hat{Q})^{-1} \in \mathbf{S} \tag{23}$$

a sufficient condition for which is that

$$|\Delta(e^{j\omega})| \leq \frac{1}{|\hat{Q}(e^{j\omega})|}, \ \forall\omega \in [-\pi,\pi] \tag{24}$$

Clearly then, a robust control can be designed on the basis of a nominal transfer function model $\hat{P}(q)$ and an upper bound on the model error as a function of frequency.

It is convenient to define the *set of uncertainty*

$$\mathbf{P}(\hat{P},\delta) = \{P(q) \in \mathbf{S} : |P(e^{j\omega}) - \hat{P}(e^{j\omega})| \leq \delta(\omega), \ \forall\omega \in [-\pi,\pi]\} \tag{25}$$

Thus, for any function $\delta(\omega) > |\Delta(e^{j\omega})|$, we have $P \in \mathbf{P}(\hat{P},\delta)$. Consequently, any control $\hat{C}$ which is designed to stabilize $\hat{P}$ and simultaneously satisfy $\delta(\omega) \leq 1/|\hat{Q}(e^{j\omega})|$, will also stabilize the true plant $P = \hat{P} + \Delta$.

3.2 Modal Truncation

Assume that the transfer function estimate $\hat{P}(q)$ is an estimate of the first n modes of the LSS. Typically n is not too large, say $n \approx 20$, and of course $n \ll \ell$. Hence, $\hat{P}(q)$ is an estimate of

$$P_n(q) = \sum_{k=1}^{n}(1-q^{-1})\mathcal{Z}\{\frac{1}{s}\frac{\beta_k}{s^2+2\zeta_k\Omega_k s+\Omega_k^2}\} \tag{26}$$

It is convenient to express $P_*(q)$ as the rational function

$$P_*(q) = \frac{B(q,\theta_*)}{A(q,\theta_*)} \tag{27}$$

where

$$\begin{aligned} A(q,\theta_*) &= 1 + a_1^* q^{-1} + \cdots + a_{2n}^* q^{-2n} \\ B(q,\theta_*) &= b_1^* q^{-1} + \cdots + b_{2n}^* q^{-2n} \\ \theta_*{}^T &= [a_1^* \cdots a_{2n}^* \; b_1^* \cdots b_{2n}^*] \end{aligned} \tag{28}$$

The transfer function coefficients $\theta_* \in \mathbf{R}^{4n}$ are uniquely determined by the modal parameters $\{\Omega_k, \beta_k, \zeta_k : k = 1, \ldots, n\}$. The converse is not true unless the sampling frequency is large enough, i.e., if $\pi > \Omega_n/\Omega_{sa}$.

By analogy with the above desription of $P_*(q)$, the estimate $\hat{P}(q)$ can be expressed as

$$\hat{P}(q) = \frac{B(q,\hat{\theta})}{A(q,\hat{\theta})} \tag{29}$$

where $\hat{\theta} \in \mathbf{R}^{4n}$ is an estimate of θ_*. Model error is then

$$\Delta(q) = P_*(q) - \hat{P}(q) + \tilde{P}(q) \tag{30}$$

where $\tilde{P}(q)$ is the *residual mode* transfer function given by

$$\tilde{P}(q) = \sum_{k=n+1}^{\ell} (1 - q^{-1}) \mathcal{Z}\{\frac{1}{s} \frac{\beta_k}{s^2 + 2\zeta_k \Omega_k s + \Omega_k^2}\} \tag{31}$$

If $\hat{\theta} \approx \theta_*$, then the n-mode estimate $\hat{P}(q)$ is close to $P_*(q)$, and $\Delta(q) \approx \tilde{P}(q)$. Thus, a compensator $\hat{C}(q)$ which stabilizes $\hat{P}(q)$ will also stabilize the true system $P_0(q)$ provided that

$$|\hat{Q}(e^{j\omega})\tilde{P}(e^{j\omega})| < 1, \; \forall \omega \in [-\pi, \pi] \tag{32}$$

To simplify the discussion to follow suppose that the sampling frequency Ω_{sa} is sufficiently large in the sense that $\Omega_n/\Omega_{sa} \ll \pi$. Otherwise anti- aliasing filters have to be included, which although they may ultimately be required, the discussion here would be obscured. In effect, we are choosing an arbitrarily large sampling frequency.

Now, for $\Omega_n/\Omega_{sa} < \omega \leq \pi$, the control can be designed so that $|\hat{Q}(e^{j\omega})|$ is as small as required. For $\omega \ll \Omega_n/\Omega_{sa}$, the residual modes have little effect, i.e., $|\tilde{P}(e^{j\omega})|$ is small. For frequencies near Ω_n/Ω_{sa}, some care is required in the control design. However, if the flexible structure is crafted so that the frequency gap $|\Omega_{n+1} - \Omega_n|$ is large, then most any controller which has a reasonable attenuation for frequencies beyond Ω_n/Ω_{sa} is likely to be robust with respect to the unmodeled residual modes. This latter approach is preferable even though the controller can be designed independently from the structure.

If $\hat{\theta}$ is not a good estimate of θ_*, then parameter error will be a large contributor to model error magnitude $|\Delta(e^{j\omega})|$. In this case, the uncertainty set $\mathbf{P}(\hat{P}, |\Delta|)$ is too coarse of a description, because the source of uncertainty is parametric rather than unmodeled high frequency dynamics. Thus, any compensator designed to robustly stabilize this set of uncertainty will be unnecessarily cautious. However, the resulting closed loop system may still satisfy the performance goals. As will be discussed later, it is possible to design the identification experiment to preclude large parameter errors.

4 Estimating the Set of Uncertainty

The problem examined in this section is to estimate a *set of uncertainty* $\mathbf{P}(\hat{P}, \delta)$ from the data record $\{y(t), u(t) : t = 1, \ldots, N\}$. We will describe one approach and show under what conditions the true plant belongs to the estimated set of uncertainty. In this case it follows that any contol designed to robustly stabilize any plant in the estimated set of uncertainty, will also stabilize the true plant.

4.1 Parametric Model Estimation

We begin by first constructing a parametric estimate of the transfer function $P_0(q)$ using the prediction error formulation in Ljung(1985,1987). Thus, the parameter estimator is given by

$$\begin{aligned} \hat{\theta}_\alpha &= \arg\min_{\theta \in \mathcal{D}} V_N(\theta, \alpha) \\ V_N(\theta, \alpha) &= \frac{1}{N} \sum_{t=1}^{N} [L_\alpha(q)\varepsilon(t, \theta)]^2 \end{aligned} \tag{33}$$

where $\mathcal{D}$ is a subset of $\mathbf{R}^p$ and $\alpha \in \mathcal{A}$ is a set of *auxiliary parameters* which characterizes a stable filter $L_\alpha(q)$, where $\mathcal{A}$ is a subset of $\mathbf{R}^m$. The purpose of the auxiliary parameter set will be desribed in the sequel. The sequence $\varepsilon(t, \theta)$ is referred to as the *prediction error* and $L_\alpha(q)\varepsilon(t, \theta)$ as the *filtered prediction error*. Except for the auxiliary parameters α, the estimator is the usual least squares estimator.

Following Ljung(1985,1987), the prediction error is obtained from the *parametric model set*

$$y(t) = P(q, \theta)u(t) + H(q, \theta)e(t), \ \theta \in \mathcal{D} \subset \mathbf{R}^p \tag{34}$$

where $e(t)$ is an unpredictable, but bounded, function. Thus, the prediction error becomes

$$\varepsilon(t, \theta) = H^{-1}(q, \theta)[y(t) - P(q, \theta)u(t)] \tag{35}$$

The only resriction on the model set is that the *predictor*, i.e., the map $(y, u) \mapsto \varepsilon$, defined implicitly above, is stable. Thus, we require that the operators $H^{-1}(q, \theta)$ and $H^{-1}(q, \theta)P(q, \theta)$ are stable, which defines the elements of the set $\mathcal{D}$. The reason for this restriction is that despite unknown initial conditions, $\limsup_{t \to \infty} \varepsilon(t, \theta) = e(t)$, which is obviously the best the predictor can do considering the unpredictability of $e(t)$.

4.1.1 Least Squares and Linear Regression

The model set we will use here is the *equation error* model set

$$A(q, \theta)y(t) = B(q, \theta)u(t) + e(t) \tag{36}$$

where $A(q, \theta)$ and $B(q, \theta)$ are polynomials in q^{-1} whose coefficients are the elements of θ, that is, let

$$\begin{aligned} A(q, \theta) &= 1 + a_1 q^{-1} + \cdots + a_{2n} q^{-2n} \\ B(q, \theta) &= b_1 q^{-1} + \cdots + b_{2n} q^{-2n} \\ \theta^T &= [a_1 \cdots a_{2n} \ b_1 \cdots b_{2n}] \end{aligned} \tag{37}$$

This formulation coincides with (27). Hence,

$$\begin{aligned} P(q,\theta) &= \frac{B(q,\theta)}{A(q,\theta)} \\ H(q,\theta) &= \frac{1}{A(q,\theta)} \end{aligned} \tag{38}$$

and the predictor becomes

$$\varepsilon(t,\theta) = A(q,\theta)y(t) - B(q,\theta)u(t) \tag{39}$$

which is obviously stable, because $A(q,\theta)$ and $B(q,\theta)$ are stable for any $\theta \in \mathbf{R}^{4n}$, thus, $\mathcal{D} = \mathbf{R}^{4n}$. The prediction error can also be written as the *linear regression*

$$\varepsilon(t,\theta) = y(t) - \theta^T\phi(t) \tag{40}$$

where $\phi(t) \in \mathbf{R}^{4n}$ is the *regressor* given by

$$\phi^T(t) = [-y(t-1)\cdots - y(t-2n)\; u(t-1)\cdots u(t-2n)] \tag{41}$$

In this case, (33) has the well known closed form solution

$$\hat{\theta}_\alpha = R_\alpha^{-1} b_\alpha \tag{42}$$

where

$$\begin{aligned} R_\alpha &= \frac{1}{N}\sum_{t=1}^{N}[L_\alpha(q)\phi(t)][L_\alpha(q)\phi(t)]^T \\ b_\alpha &= \frac{1}{N}\sum_{t=1}^{N}[L_\alpha(q)\phi(t)][L_\alpha(q)y(t)] \end{aligned} \tag{43}$$

provided that R_α^{-1} exists. A sufficient condition is that $L_\alpha(q)\phi(t)$ is persistently exciting and the length of the data record N is sufficiently large.

DEFINITION

A sequence $f(t) \in \mathbf{R}^p$ is said to be persistently exciting if there is a positive constant β and a positive integer M such that for all $\tau \in [0,\infty)$

$$\min_{i\in[1,p]} \lambda_i \left\{ \frac{1}{M}\sum_{t=\tau+1}^{\tau+M} f(t)f^T(t) \right\} \geq \beta \tag{44}$$

Clearly if $L_\alpha(q)\phi(t)$ is persistently exciting and $N \geq M$, then R_α^{-1} exists.

4.1.2 Estimating the Nominal Transfer Function

Having found $\hat{\theta}_\alpha$, form the parametric plant transfer function estimate, denoted by $\hat{P}_\alpha(q)$, using the equation error model, i.e.,

$$\hat{P}_\alpha(q) = \frac{B(q,\hat{\theta}_\alpha)}{A(q,\hat{\theta}_\alpha)} \tag{45}$$

Thus, we obtain the *family* of parametric models

$$\{\hat{P}_\alpha(q), \alpha \in \mathcal{A}\} \tag{46}$$

Observe that the true plant $P_0(q)$ belongs to *every* member of the family of sets of uncertainty

$$\{\mathbf{P}(\hat{P}_\alpha, |\Delta_\alpha|), \alpha \in \mathcal{A}\} \tag{47}$$

where the true model error is

$$\Delta_\alpha(q) = \hat{P}_\alpha(q) - P_0(q) \tag{48}$$

Using the data record $\{y(t), u(t) : t = 1, \ldots, N\}$, we now seek to find the family of sets of uncertainty

$$\{\mathbf{P}(\hat{P}_\alpha, \delta_\alpha), \alpha \in \mathcal{A}\} \tag{49}$$

with the property that

$$\delta_\alpha(\omega) \geq |\Delta_\alpha(e^{j\omega})|, \forall \omega \in [-\pi, \pi], \forall \alpha \in \mathcal{A} \tag{50}$$

which is sufficient to guaranty that for all $\alpha \in \mathcal{A}$ we have $P_0 \in \mathbf{P}(\hat{P}_\alpha, \delta_\alpha)$.

4.1.3 Bias in Least Squares Parameter Estimate

The parameter error $\hat{\theta}_\alpha - \theta_*$, or *bias*, is a result of unmodeled dynamics and disturbances. To compute the bias, observe that (14) can be expressed as

$$P_0(q) = \frac{B(q, \theta_*)}{A(q, \theta_*)} + \tilde{P}(q) \tag{51}$$

Hence, it follows that (15) can be written as

$$y(t) = \theta_*{}^T \phi(t) + \varepsilon(t, \theta_*) \tag{52}$$

where the prediction error is

$$\varepsilon(t, \theta_*) = A(q, \theta_*)[\tilde{P}(q)u(t) + d(t)] \tag{53}$$

Substituting this expression into (42) gives the bias as

$$\hat{\theta}_\alpha - \theta_* = R_\alpha^{-1} \tilde{b}_\alpha \tag{54}$$

where

$$\tilde{b}_\alpha = \frac{1}{N} \sum_{t=1}^{N} [L(q)\phi(t)][L(q)\varepsilon(t, \theta_*)] \tag{55}$$

By carefully choosing the data filter $L(q)$ and the input spectrum $S_{uu}(\omega)$, it is clear that the bias can be made small. Hence, it is possible to design the identification experiment so that $\hat{P}_\alpha(q) \approx B(q, \theta_*)/A(q, \theta_*)$, and the inherent conservatism in the set of uncertainty $\mathbf{P}(\hat{P}_\alpha, \delta_\alpha)$ due to parameter bias is insignificant. Issues of experiment design will not be pursued here. The interested reader is referred to Ljung(1987) and the references therein.

4.2 Non-Parametric (Spectral) Estimation

The approach we propose for estimating the model error is to use standard methods of non-parametric transfer function estimation based on spectral estimation, e.g., Jenkins and Watts(1968), Ljung(1985,1987). We will assume that $u(t)$ and $d(t)$ are zero-mean sequences with spectral densities $S_{uu}(\omega)$ and $S_{dd}(\omega)$, and cross- spectral density $S_{du}(\omega)$. These are all defined analogously with (17). Assume that the input and disturbance are uncorrelated, i.e.,

$$S_{du}(\omega) = 0 \tag{56}$$

This situation arises, for example, when the LSS is operating without a stabilizing feedback, that is, in open-loop. Recall that the LSS is open- loop stable. (The proceedure described below can be modified to account for a stabilizing feedback. A brief discussion appears later in the sequel.)

Using the parametric transfer function estimate $\hat{P}_\alpha(q)$, form the corresponding *output error*, i.e.,

$$\begin{aligned} \eta_\alpha(t) &= y(t) - \hat{P}_\alpha(q)u(t) \\ &= \Delta_\alpha(q)\bar{u}(t) + d(t) \end{aligned} \tag{57}$$

where

$$\Delta_\alpha(q) = P_0(q) - \hat{P}_\alpha(q) \tag{58}$$

is the true model error, for which we now seek an estimate based on the data records

$$\{\eta_\alpha(t), u(t) : t = 1, \ldots, N\} \tag{59}$$

for each $\alpha \in \mathcal{A}$.

Using the same notation as above, the cross-spectral density between the output error and the input is given by

$$S_{\eta_\alpha u}(\omega) = \Delta_\alpha(e^{j\omega})S_{uu}(\omega) \tag{60}$$

The standard non-parametric frequency domain estimate for $\Delta_\alpha(e^{j\omega})$ is then

$$\hat{\Delta}_\alpha(\omega) = \frac{\hat{S}_{\eta_\alpha u}(\omega)}{\hat{S}_{uu}(\omega)} \tag{61}$$

where $\hat{S}_{\eta_\alpha u}(\omega)$ and $\hat{S}_{uu}(\omega)$ are spectral density estimates obtained from the finite data record (59). There are many specific ways to generate spectral density estimates. Following Ljung(1985,1987) or Jenkins and Watts(1968) we use the *smoothed* spectral estimators:

$$\begin{aligned} \hat{S}_{\eta_\alpha u}(\omega) &= \sum_{\tau=-N+1}^{N-1} \lambda_\gamma(\tau)\hat{R}_{\eta_\alpha u}(\tau)e^{-j\omega\tau} \\ \hat{S}_{uu}(\omega) &= \sum_{\tau=-N+1}^{N-1} \lambda_\gamma(\tau)\hat{R}_{uu}(\tau)e^{-j\omega\tau} \end{aligned} \tag{62}$$

where the correlation estimates are

$$\begin{aligned} \hat{R}_{\eta_\alpha u}(\tau) &= \frac{1}{N}\sum_{t=1}^{N-|\tau|} \eta_\alpha(t)u(t+|\tau|) \\ \hat{R}_{uu}(\tau) &= \frac{1}{N}\sum_{t=1}^{N-|\tau|} u(t)u(t+|\tau|) \end{aligned} \tag{63}$$

The function $\lambda_\gamma(\tau)$ is referred to as the *lag window of length* γ, where typically $\lambda_\gamma(\tau) = 0$ whenever $|\tau| > \gamma$. The above spectral estimates can be equivalently expressed as the following frequency domain convolutions:

$$\begin{aligned} \hat{S}_{\eta_\alpha u}(\omega) &= \int_{-\pi}^{\pi} W_\gamma(\omega - \sigma)I^N_{\eta_\alpha u}(\sigma)\,d\sigma \\ \hat{S}_{uu}(\omega) &= \int_{-\pi}^{\pi} W_\gamma(\omega - \sigma)I^N_{uu}(\sigma)\,d\sigma \end{aligned} \tag{64}$$

where $W_\gamma(\omega)$ is the *spectral window*, defined as the Fourier transform of the lag window, i.e.,

$$W_\gamma(\omega) = \sum_{\tau=-N+1}^{N-1} \lambda_\gamma(\tau) e^{-j\omega t} \tag{65}$$

and where $I^N_{\eta_\alpha u}(\omega)$ and $I^N_{uu}(\omega)$ are the *periodograms* given by

$$\begin{aligned} I^N_{\eta_\alpha u}(\omega) &= \frac{1}{2\pi N}\left(\sum_{t=1}^{N}\eta_\alpha(t)e^{-j\omega t}\right)\overline{\left(\sum_{t=1}^{N}u(t)e^{-j\omega t}\right)} \\ I^N_{uu}(\omega) &= \frac{1}{2\pi N}\left|\sum_{t=1}^{N}u(t)e^{-j\omega t}\right|^2 \end{aligned} \tag{66}$$

By combining the above expressions, the frequency domain estimate of model error can be compactly expressed as

$$\hat{\Delta}_\alpha(\omega) = \frac{\int_{-\pi}^{\pi} W_\gamma(\omega-\sigma) I^N_{\eta_\alpha u}(\sigma)\, d\sigma}{\int_{-\pi}^{\pi} W_\gamma(\omega-\sigma) I^N_{uu}(\sigma)\, d\sigma} \tag{67}$$

Observe that as γ increases the spectral window becomes more narrow. In addition, the spectral window is usually characterized by the following properties:

$$\begin{aligned} &\int_{-\pi}^{\pi} W_\gamma(\omega)d\omega = 1 \qquad \int_{-\pi}^{\pi} \omega W_\gamma(\omega)d\omega = 0 \\ &\int_{-\pi}^{\pi} \omega^2 W_\gamma(\omega)d\omega = M(\gamma) \quad \int_{-\pi}^{\pi} W^2_\gamma(\omega)d\omega = \tfrac{1}{2\pi}K(\gamma) \end{aligned} \tag{68}$$

where as γ increases, $M(\gamma)$ decreases and $K(\gamma)$ increases. Under these conditions, it can be shown [see, e.g., Jenkins and Watts(1968) or Ljung(1985,1987)], that for large N, large γ, and small $K(\gamma)/N$:

$$|\hat{\Delta}_\alpha(\omega) - \Delta_\alpha(e^{j\omega})|^2 \approx M^2(\gamma)|R_\alpha(\omega)|^2 + \frac{K(\gamma)}{N}\frac{S_{dd}(\omega)}{S_{uu}(\omega)} \tag{69}$$

where

$$R_\alpha(\omega) = \frac{1}{2}\Delta''_\alpha(e^{j\omega}) + \Delta'_\alpha(e^{j\omega})\frac{S'_{uu}(\omega)}{S_{uu}(\omega)} \tag{70}$$

with $'$ and $''$ denoting differentiation with respect to ω, once and twice, respectively. Hence, for some large N, large γ, and small $K(\gamma)/N$, the error $|\hat{\Delta}_\alpha(\omega) - \Delta_\alpha(e^{j\omega})|$ can be made arbitrarily small. This fact justifies the proposed uncertainty modeling scheme for on-line robust control design. Specifically, let $T(P,C)$ denote the closed-loop system corresponding to the configuration of Figure ??, let $\mathbf{T}_*$ denote the set of all acceptable closed-loop transfer matrices, let P_0 denote the true plant, and for each $\alpha \in \mathcal{A}$, let $\mathbf{P}(\hat{P}_\alpha, \delta_\alpha)$ denote the set of uncertainty defined by

$$\mathbf{P}(\hat{P}_\alpha, \delta_\alpha) = \{P(q) \in \mathbf{S} : |P(e^{j\omega}) - \hat{P}_\alpha(e^{j\omega})| \le \delta_\alpha(\omega)|\} \tag{71}$$

We can now state:

THEOREM 2

Suppose that for all $\alpha \in \mathcal{A}$ there is a function

$$\delta_\alpha(\omega) > |\hat{\Delta}_\alpha(\omega)|, \ \forall \omega \in [-\pi, \pi] \tag{72}$$

If there is an $\alpha_* \in \mathcal{A}$ and a corresponding compensator $C_{\alpha_*}(q)$, such that

$$T(P, C_{\alpha_*}) \in \mathbf{T}_*, \ \forall P \in \mathbf{P}(\hat{P}_{\alpha_*}, \delta_{\alpha_*}) \tag{73}$$

then, for some large N, large γ, and small $K(\gamma)/N$,

$$T(P_0, C_{\alpha_*}) \in \mathbf{T}_* \tag{74}$$

Theorem 2 provides conditions under which the true plant is a member of the estimated set of uncertainty. Hence, any controller which robustly stabilizes the set of uncertainty will also stabilize the true plant. In this context robust stabilization means that the closed-loop system $T(P, C_{\alpha_*}) \in \mathbf{T}_*$ for all P in the set of uncertainty.

The difficulty in applying the theorem is to properly select N, γ, and most importantly, the function $\delta_\alpha(\omega)$. A natural choice for the latter is

$$\delta_\alpha(\omega) = [1 + k(\omega)]|\hat{\Delta}_\alpha(\omega)| \tag{75}$$

where $k(\omega) > 0$. Hence, $\delta_\alpha(\omega) \geq |\Delta_\alpha(e^{j\omega})|, \forall \alpha \in \mathcal{A}$ if

$$\sigma_\alpha(\omega) < \frac{k(\omega)}{1 + k(\omega)}, \ \forall \alpha \in \mathcal{A} \tag{76}$$

where

$$\sigma_\alpha(\omega) = \frac{1}{|\Delta_\alpha(e^{j\omega})|} \left(M^2(\gamma)|R_\alpha(\omega)|^2 + \frac{K(\gamma)}{N} \frac{S_{dd}(\omega)}{S_{uu}(\omega)} \right)^{1/2} \tag{77}$$

For large γ, typical spectral window characteristics behave like

$$M(\gamma) \approx \frac{M_0}{\gamma^2}, \ \ K(\gamma) \approx K_0 \gamma \tag{78}$$

Hence, for large γ,

$$\sigma_\alpha \approx \frac{1}{|\Delta_\alpha(e^{j\omega})|} \left(\frac{M_0^2}{\gamma^4}|R_\alpha(\omega)|^2 + K_0 \frac{\gamma}{N} \frac{S_{dd}(\omega)}{S_{uu}(\omega)} \right)^{1/2} \tag{79}$$

Observe that if $S_{uu}(\omega)$ is more or less constant for $\omega \leq \Omega_n/\Omega_{sa}$ and attenuates rapidly thereafter, which is usually the case, then $R_\alpha(\omega) \approx (1/2)\Delta_\alpha''(\omega)$. Thus, for some large γ, $\delta_\alpha(\omega) \geq |\Delta_\alpha(e^{j\omega})|$, provided that

$$\frac{M_0}{2\gamma^2} \left| \frac{\Delta_\alpha''(e^{j\omega})}{\Delta_\alpha(e^{j\omega})} \right| < \frac{k(\omega)}{1 + k(\omega)} \text{ and } \frac{\gamma}{N} \frac{S_{dd}(\omega)}{S_{uu}(\omega)} \ll 1 \tag{80}$$

4.2.1 Summary of Uncertainty Estimation Procedure

The following steps summarize the above procedure for estimating model uncertainty:

Step 1 Given a preselected filter parameter $\alpha \in \mathcal{A}$, form the filtered prediction error, and then solve for $\hat{\theta}_\alpha$ from (33).

Step 2 Using the parametric transfer function estimate $P(q, \hat{\theta}_\alpha)$, form the output error data record (59), and then calculate the frequency domain model uncertainty estimate $\hat{\Delta}_\alpha(\omega)$ from (67), using standard spectral estimation procedures.

Step 3 Design a *robust* control based on the plant being a member of the estimated set of uncertainty $\mathbf{P}(\hat{P}_\alpha, k|\hat{\Delta}_\alpha|)$, as defined in (71) for some function $k(\omega) > 0, \forall \omega \in [-\pi, \pi]$.

Step 4 If, for some $\alpha_* \in \mathcal{A}$, there is a feedback compensator $C_{\alpha_*}(q)$, such that the closed-loop systems in the set $\{T(P, C_{\alpha_*}), \forall P \in \mathbf{P}(\hat{P}_{\alpha_*}, \delta_{\alpha_*})\}$ satisfy performance objectives, then *implement* the compensator. Otherwise, go to Step 5.

Step 5 Select a new value for the filter parameters $\alpha \in \mathcal{A}$ and go to Step 1. If all filter parameter values in $\mathcal{A}$ have been exhausted, then implement that compensator in the set $\{C_\alpha, \alpha \in \mathcal{A}\}$ which produces the best closed-loop performance, i.e., the nearest to the performance objectives.

4.3 Closed-Loop Uncertainty Estimation

Suppose that the LSS system

$$y(t) = P_0(q)u(t) + d(t) \tag{81}$$

is operating in closed-loop with the stabilizing feedback

$$u(t) = r(t) - C_0(q)y(t) \tag{82}$$

The least-squares parametric procedure for obtaining $\hat{P}_\alpha$ will still work, but the spectral approach for obtaining the model error estimate $\hat{\Delta}_\alpha$ needs to be modified. The reasoning is as follows: let $r(t) = 0$ and suppose that the spectral estimates $\hat{S}_{\eta_\alpha u}(\omega)$ and $\hat{S}_{uu}(\omega)$ are very close to the true spectra $S_{\eta_\alpha u}(\omega)$ and $S_{uu}(\omega)$, respectively. We then have

$$\hat{\Delta}_\alpha(\omega) - \Delta_\alpha(e^{j\omega}) \approx \frac{S_{\eta u}(\omega)}{S_{uu}(\omega)} - \Delta_\alpha(e^{j\omega}) \tag{83}$$

$$= -\left[\frac{1}{C_0(e^{j\omega})} + P_0(e^{j\omega})\right] \tag{84}$$

which can be an arbitrarily bad estimate. To correct this difficulty, suppose that $r(t)$, which is available to the user, is selected so that it is uncorrelated with the disturbance $d(t)$ and its spectrum is much larger than the disturbance spectrum, i.e., $S_{rd}(\omega) = 0$ and $S_{rr}(\omega) \gg S_{dd}(\omega)$. If the spectral estimates are close to their true values then

$$\hat{\Delta}_\alpha(\omega) - \Delta_\alpha(e^{j\omega}) \approx \frac{S_{\eta u}(\omega)}{S_{uu}(\omega)} - \Delta_\alpha(e^{j\omega}) \tag{85}$$

$$= O\left(\frac{S_{dd}(\omega)}{S_{rr}(\omega)}\right) \tag{86}$$

Thus, by careful external input selection (experiment design), it is possible to use the uncertainty estimation procedure also in closed loop with a nominal stabilizing controller.

5 Example of Uncertainty Estimation

In this section the uncertainty estimation procedure is applied to data obtained from the laser pointing experiment described in Walker et al.(1984). A schematic drawing of the apparatus is shown in Figure 4.

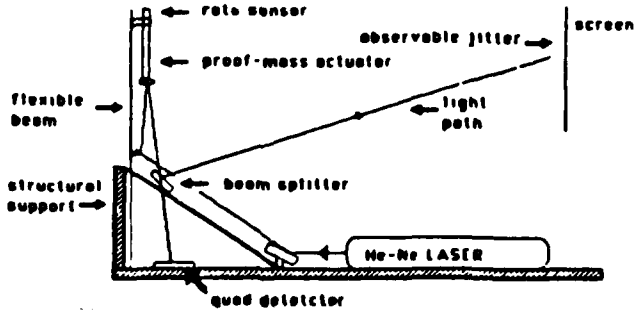

Figure 4: Laser pointing and control experiment

The objective of the experiment is to control the jitter of the laser beam. The single actuator consists of a pivoted proof-mass which exerts a reaction force on the flexible beam whenever

the proof-mass is moved by an applied control current through the armature. As the flexible structure vibrates, the laser beam changes its angular direction. A quadrature detector, mounted on the structural support, registers the laser beam position, as long as it is in the field-of-view of the detector. The laser beam strikes a mirror on the flexible structure and is reflected by another mirror mounted on the proof- mass actuator. The resulting beam is split in two by a beam-splitter, with one ray going to the quadrature detector and the other to a screen where the jitter is magnified for visual inspection. The mass of the actuator is greator than the mass of the flexible structure, thereby insuring a significant interaction between the actuator and the flexible structure.

The problem is to control the jitter from 4 Hz to 20 Hz. The input $u(t)$ is chosen as a sine-sweep lasting about 16 seconds, sweeping from 4 Hz to 20 Hz and sampled at 51.2 Hz. The number of data samples is $N = 1024$.

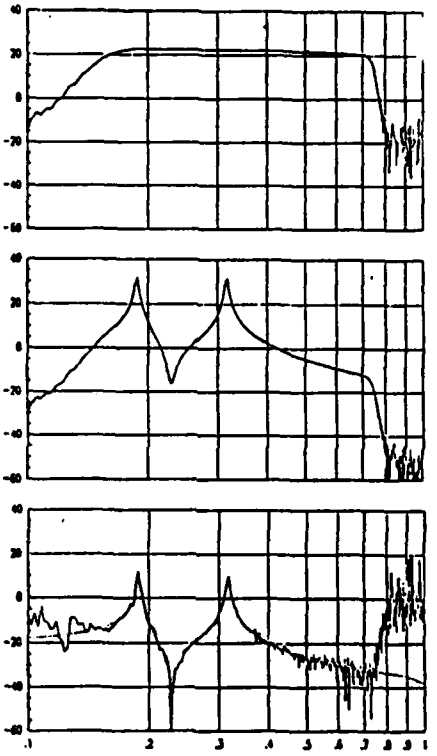

Figure 5: (a) DFT(u), (b) DFT(y), (c) ETFE and $P_0(e^{j\omega})$

Figure 5 shows the discrete-Fourier-transform (DFT) of the input and output, as well the ratio of output DFT to input DFT, referred to as the *emperical transfer function estimate* (ETFE), compared to the "true" transfer function $P_0(e^{j\omega})$. Since the data is taken from a physical device, the true system dynamics are not really known. What we refer to as the "true" system is an 8th order equation error model obtained from the data by using the least squares estimate (42) with $n = m = 8$. This gives

$$
\begin{aligned}
P_0(q) &= \frac{B_0(q,\theta)}{A_0(q,\theta)} \\
B(q,\theta) &= -.0044q^{-1} + .0016q^{-2} - .056q^{-3} - .015q^{-4} \\
&\quad + .013q^{-5} - .011q^{-6} - .033q^{-7} - .027q^{-8} \\
A_0(q,\theta) &= 1 - 1.11q^{-1} + .54q^{-2} + .47q^{-3} + .77q^{-4} \\
&\quad - .17q^{-5} + .08q^{-6} + .40q^{-7} + .0057q^{-8}
\end{aligned}
$$

The parametric model is chosen as a 4th-order equation error model as in (36), where

$$\begin{aligned} A(q,\theta) &= 1 + a_1 q^{-1} + \cdots + a_4 q^{-4} \\ B(q,\theta) &= b_1 q^{-1} + \cdots + b_4 q^{-4} \\ \theta^T &= [a_1, \ldots, a_4, b_1, \ldots, b_4] \end{aligned}$$

The auxiliary filter parameter set is the 4-tuple,

$$\mathcal{A} = \{0, 1, 2, 3\}$$

such that the corresponding filters have the following properties:

$L_0(q) = 1$, the "natural" filter. .

$L_1(q) =$ 8th-order Butterworth with passband $[.11\pi, .67\pi]$.

$L_2(q) =$ 8th-order Butterworth with passband $[.11\pi, .40\pi]$.

$L_3(q) =$ 16th-order Butterworth with passband $[.15\pi, .40\pi]$.

The effect of each filter is shown, respectively, in Figures 6 to 9. Each figure shows 3 plots: the two top ones showing gain and phase of the estimated plant (the dashed lines) compared to the true plant (the solid lines), and the bottem plot comparing magnitudes of the model error estimate (the dashed line) from (67) with the true model error (the solid line).

The top and bottem plots also show the magnitude of the passband filter used to form the filtered prediction error. Observe that the model error estimates are quite accurate in the frequency range [.15, .70], which is the same range where the input has a nice flat DFT. The model error estimates were obtained using a rectangular window, and no attempt was made here to adjust the lag window width to achieve a better resolution. The best model estimate is clearly for the filter $L_3(q)$, particularly over the passband of interest.

These results give confidence to the uncertainty model estimation procedure proposed here, because despite the fact the some of the estimated transfer functions are quite poor, the model uncertainty estimate is very good. Hence, a robust control designed on the basis of the estimated set of uncertainty would not de-stabilize the true system. The next step is to continue with the above example, for example, and incorporate a specific robust design control design method.

6 Concluding Remarks

A method for estimating the set of uncertainty of a plant transfer function has been proposed and analyzed. Theorem 2 provides analytic justification that the true plant can, under suitable conditions, be guaranteed to be in the estimated set of uncertainty. An example is presented which supports the theory. One of the restrictions in this paper is that the plant is stable and no feedback is present during the experiment. The results, however, can be extended to the case of an unstable plant operating in closed-loop with a stabilizing feedback compensator.

Although in this paper we restrict the auxiliary parameters to essentially index data filters, the concept can be broadened. For example, the filters can be chosen differently for the input and output data. In addition, the auxiliary parameters can index various inputs which have different spectral content and the choice made by using a local optimization on the auxiliary parameters. The index parameters are thus choices which can be made to tailor the identification experiment in accordance with the use of the model. Off- line procedures for experiment design are discussed,

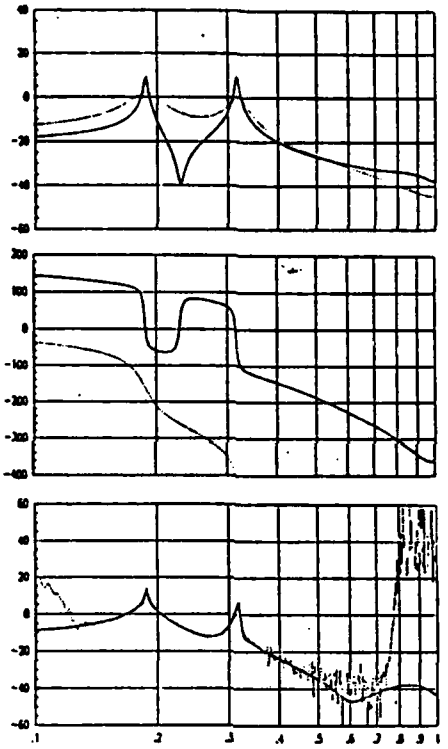

Figure 6: (a) gain, (b) phase, (c) model error magnitude

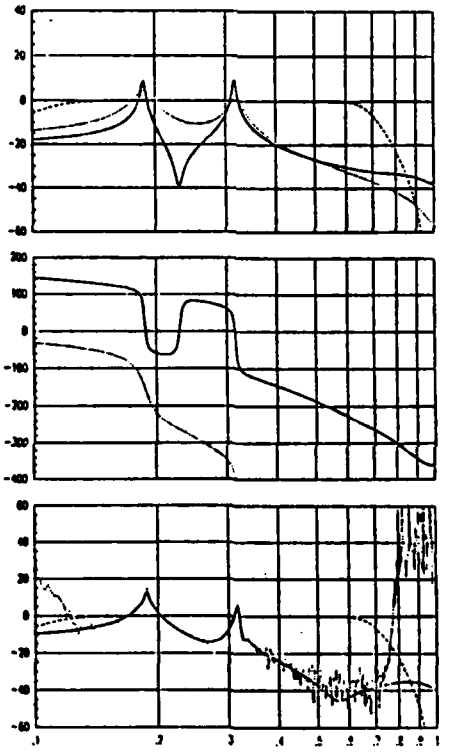

Figure 7: (a) gain, (b) phase, (c) model error magnitude

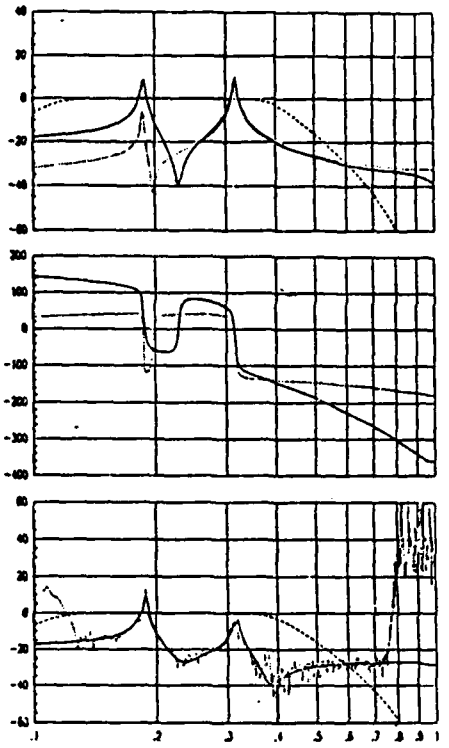

Figure 8: (a) gain, (b) phase, (c) model error magnitude

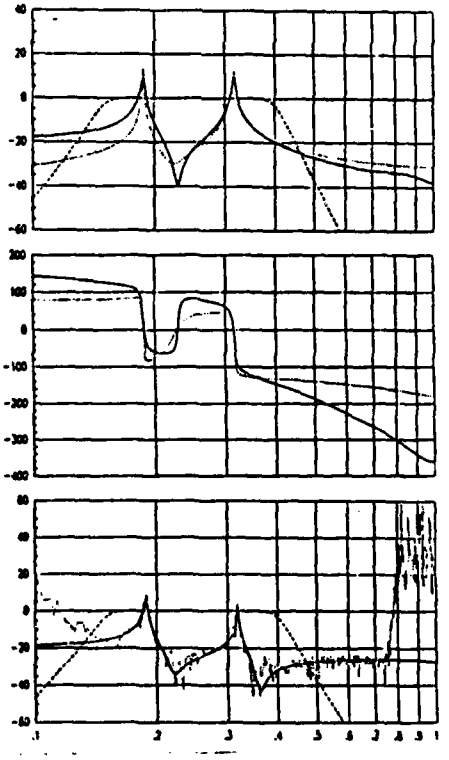

Figure 9: (a) gain, (b) phase, (c) model error magnitude

for example, in Ljung(1987) or Goodwin and Payne(1977). The issue addressed here, which might be called "on-line experiment design", remains to be studied and further developed.

The results of this paper raise some questions, like: why not use non-parametric spectral estimation to obtain the transfer function estimate in the first place? The answer is partly to be found in Ljung(1985,1987), where it is shown, as might be expected, that the parametric techniques have a smaller variance (asymptotically) as a function of data length. Here, we are using the parametric model for the critical frequency range where control is needed, whereas a cruder spectral analysis is used for the frequency range over which control is not as important. Another question: why not just increase the model order, and hence, stay with parametric methods? This approach is taken in Wahlberg(1986), where after identification a model reduction is performed. The problem is that we now have *one* plant estimate and no set of uncertainty, i.e., the estimate is assumed to be sufficiently accurate. Hopefully it is, but this can only be established by an *a priori* analysis, i.e., there is no on-line estimate of the set of uncertainty. It may be that such an estimate is unnecessary for a particular problem, provided that the experiment has been appropriately designed.

7 References

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, (1986),*Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

J.C. Doyle and C.C. Chu (1986),"Robust Control of Multivariable and Large Scale Systems", Final Tech. Report under AFOSR Contract F49620-84-C-0088, Honeywell Systems and Research Center, Minn., MN.

B.A. Francis and G. Zames (1984), "On H_∞-Optimal Sensitivity Theory for SISO Feedback Systems", *IEEE Trans. on Aut. Control*, Vol. 29, pp. 9-16.

G.C. Goodwin and R.L. Payne, (1977), *Dynamic System Identification: Experiment Design and Data Analysis*, Academic Press, New York.

G.M. Jenkins and D. G. Watts, (1968) *Spectral Analysis and its Applications*, Holden-Day, San Francisco.

R.L. Kosut, (1986), "Adaptive Calibration: An Approach to Uncertainty Modeling and On-Line Robust Control Design", *Proc. 25th IEEE CDC*, Athens, Greece, Dec. 1986.

R.L. Kosut, (1987), "Adaptive Uncertainty Modeling: On-Line Robust Control Design", *Proc. 1987 ACC*, Minneapolis, MN, June 10-12, 1987.

L. Ljung, (1985), "On the Estimation of Transfer Functions", *Automatica*, Vol. 21, No. 6, Nov.

L. Ljung, (1987), *System Identification: Theory for the User*, Prentice Hall, New Jersey.

M.G.Safonov, A.J.Laub, and G.L.Hartman(1981),"Feedback Properties of Multivariable Systems: The Role and Use of the Return Difference Matrix", *IEEE Trans. Aut. Contr.*, **Vol. 26**(1):47-65, Feb. 1981.

M. Vidyasagar (1985), *Control system Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

B. Wahlberg (1986), "On Model Reduction in System Identification", *Proc. ACC*, Seattle, WA, June.

Adaptive Control via Finite Modeling and Robust Control

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Abstract

This papers examines an adaptive control scheme which involves the selection of a fixed controller out of a finite collection, where each controller is robustly designed to account for the true plant being in a not necessarily small set of uncertainty. There are no adaptive "parameters" in the conventional sense where adjustments are made to either model or controller parameters. The parameters here involve pseudo-probabilities or weights assigned to each of the finite controllers. These quantities are computed recursively from the measured data. Various adaptive selection mechanisms which depend on these pseudo-probabilities will be presented and discussed in the full paper.

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1 Introduction

Our concern is with the adaptive control of plants with both parametric (structured) uncertainty and dynamical (unstructured) uncertainty. The plant is linear, and subject to disturbance and measurement noise. Knowing the value of the uncertain parameters allows for the design of a robust controller to cope with the unstructured uncertainty. Moreover, there is no robust controller, i.e., one that is linear-time-invariant, that will handle the entire spread of structured and unstructured uncertainty. Hence, there is the need for an adaptive control.

More precisely, let $\mathcal{P}$ denote the family of plants given by

$$\mathcal{P} = \{P(z) = P(z,\alpha)[1 + \Delta(z,\alpha)] : \alpha \in \mathcal{A},\ |\Delta((e^{j\omega}),\alpha)| \leq |\delta((e^{j\omega}),\alpha)|, \forall \omega \in [-\pi,\pi]\} \quad (1)$$

where α is the structured parameter constrained to a subset $\mathcal{A}$ of $\mathbf{R}^p$, and $\Delta(z,\alpha)$ is the unstructured uncertainty bounded by the weighting function $\delta(z,\alpha)$. Assume that there is a finite number of parameter "design" values $\alpha_1,\ldots,\alpha_N$, each in $\mathcal{A}$, such that for *any* $\alpha \in \mathcal{A}$, not necessarily equal to $\alpha_1,\ldots,\alpha_N$, there exists a controller $C_i(z)$, tuned to α_i for some i, which acheives satisfactory performance in the face of the unstructured uncertainty, and in the face of the difference between the actual value of α and the design value α_i. For example, the i-th controller $C_i(z)$ in the set $C_1(z),\ldots,C_N(z)$, provides robust performance for any plant in the family

$$\mathcal{P}_i = \{P(z) \in \mathcal{P} : \|\alpha - \alpha_i\| \leq m_i\} \quad (2)$$

where $\mathcal{P}_1,\ldots,\mathcal{P}_N$ completely cover the plant set $\mathcal{P}$, that is

$$\mathcal{P} \subset \bigcup_{i=1}^{N} \mathcal{P}_i \quad (3)$$

Our task in this paper is to explain how, when α is unknown, the controller $C_i(z)$ can be selected to control the plant. This type of adaptive controller is not like the conventional ones where either model or controller parameters are directly adjusted. Here the parameters are contained in the mechanism for switching amongst the preselected robust controllers, which is essentially a gain scheduling proceedure, but is adaptive in the sense that the schedule is being learned from the measured data. The gain schedule is usually set in advance, for example, in a flight control system the gain schedule is a predetermined function of the Mach number and aerodynamic pressure. One of the interesting possible adavantages of this method is that although the plant may have a large number of uncertain parameters (α in the above notation), it is possible that only a few controllers are required, and hence only a few parameters in the selection mechanism. Also the individual robust controllers can be based on uncertainty in physical parameters rather than canonical parameters, such as transfer function coefficients, as used in the conventional parameter estimation techniques, e.g., least squares with a linear regression model.

The idea of using preselected robust controllers as described above is due to K. Poola [1] who also proposed a particular smooth adaptive selection algorithm. Here, we examine the adaptive selection mechanism as discussed in [2] and [3], which will be described in detail in

the sequel. In [3] this scheme is referred to as a Multiple Model Adaptive Control (MMAC), but the individual controllers are not necessarily selected to be robust in the manner described above. Here we also provide an analysis of the convergence properties of the adaptive selection algorithm following the analysis in [2].

2 Structure of the Adaptive Controller

Figure 1 depicts the general adaptive set-up that we consider. The design of each controller $C_i(z)$ is not an adaptive control design, but rather, a robust control design task involving the plant family $\mathcal{P}$ and the coverings $\mathcal{P}_i$. The signals used to adaptively select the controller are the innovations sequences, or prediction errors, denoted by $\varepsilon_1(t), \ldots, \varepsilon_N(t)$, where each is obtained from the Kalman filters denoted $KF_1, \ldots, KF_N$.

Although there are many structures for the individual controllers $C_i(z)$, one structure which utilizes the prediction errors is shown in Figure 2. In this case $C_i(z)$ consists of an observer based state feedback controller together with an auxiliary signal obtained by processing the observer innovations sequence $\varepsilon_i(t)$ through a stable transfer function $Q_i(z)$. An important aspect regarding the flexibility of this controller structure is that each $C_i(z)$ is a parametrization, in terms of stable $Q_i(z)$, of all stabilizing controllers of the plant model $P(z, \alpha_i)$, see, e.g.,[4]. That is, *all* controllers which stabilize $P(z, \alpha_i)$ are obtained by letting $Q_i(z)$, referred to as the *Youla parameter*, range over all stable transfer functions.

The adaptive selection mechanism can be abrupt or smooth, and we will in the full paper examine in detail both of these choices. First, we review the results in [2] amd then provide the extension to the feedback case. The adaptation mechanism is the real object of interest in this paper. We set it up using the ideas of [2] and [3]. In broad outline, the idea is this: Design predictors (e.g., Kalman filters) which are optimal for each of the plants $P(z, \alpha_i)$, $i = 1, \ldots, N$. (Their design also requires values to be assigned to noise covariances.) We then process the innovations, or prediction errors, from each predictor and compute certain quantities, roughly, the scalar sample covariance. The index value $i_* \in [1, \ldots, N]$ whose predictor produces the smallest sample covariance is selected and the corresponding $C_{i_*}(z)$ controller is used to control the plant.

3 Review of [2]

Two issues are examined in [2]. First, when the plant is in the model set, and secondly, when it is not.

3.1 Plant in Model Set

Let a plant be drawn from the collection $\{P(z,\alpha_i) : i = 1,\ldots,N\}$, say the plant is $P(z,\alpha_{i_o})$. Suppose the plant has disturbances and measurement noise, but no exogenous input. Let N Kalman filters, each tuned to $\{P(z,\alpha_i) : i = 1,\ldots,N\}$, be connected to the plant. Let the *asymptotic* design innovation covariances be Ω_i and let $\varepsilon_i(t)$ denote the sequence obtained from the i-th filter connected to $P(z,\alpha_{i_o})$, at the point in the filter where the innovation sequence would normally be observed. Recall that the quantities $\varepsilon_i(t)$ and Ω_i are available from the Kalman filter and covariance equations, respectively. Moreover, the design covariance Ω_i is computible in adavance of the measurements. Note also that $\varepsilon_{i_o}(t)$ is actually an innovation sequence, but $\varepsilon_i(t)$ for $i \neq i_o$ is not in general. Define, for $i \neq i_o$,

$$L_i(t) = \frac{p(\alpha_i|z^t)}{p(\alpha_{i_o}|z^t)} \tag{4}$$

where z^t denotes the sequence of measurements $y(1),\ldots,y(t)$, and $p(\alpha|z^t)$ are the *a posteriori probabilities* which, assuming gaussian distributions, are sequentially computible from

$$p(\alpha_i|z^t) = \frac{p(\alpha_i|z^{t-1})\sqrt{\det(\Omega_i^{-1})}\exp\{-\frac{1}{2}\varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)\}}{\sum_{j=1}^N p(\alpha_j|z^{t-1})\sqrt{\det(\Omega_j^{-1})}\exp\{-\frac{1}{2}\varepsilon_j^T(t)\Omega_j^{-1}\varepsilon_j(t)\}} \tag{5}$$

It follows immediately that

$$\ln L_i(t) = \ln L_i(t-1) + \frac{1}{2}[\varepsilon_{i_o}^T(t)\Omega_{i_o}^{-1}\varepsilon_{i_o}(t) - \varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)] \tag{6}$$

Thus, as $t \to \infty$, and for all $i \neq i_o$,

$$\frac{2}{t}\ln[L_i(t)] \to -(V_i - V_{i_o}) \tag{7}$$

where for *all* i,

$$V_i = \ln\det(\Omega_i) + \mathrm{tr}(\Omega_i^{-1}\Sigma_i) \tag{8}$$

with

$$\Sigma_i = \lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t}\varepsilon_i(k)\varepsilon_i^T(k) \tag{9}$$

$$= \mathcal{E}\{\varepsilon_i(t)\varepsilon_i^T(t)\} \tag{10}$$

The expectation operator $\mathcal{E}(\cdot)$ is of course taken with respect to the noise processes. By definition of optimality,

$$V_{i_o} < V_i,\ \forall i \neq i_o \tag{11}$$

and thus, $\forall i \neq i_o$, as $t \to \infty$,

$$\frac{p(\alpha_i|z^t)}{p(\alpha_{i_o}|z^t)} \to 0 \tag{12}$$

exponentially fast with asymptotic convergence rate $\exp\{-(V_i - V_{i_o})t\}$. This means that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_o \\ 0, & i \neq i_o \end{cases} \tag{13}$$

and hence, the correct plant model is captured. Intuitively this is as expected, because the plant $P(z, \alpha_{i_o})$ is in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, and moreover, one of the Kalman filters is optimal.

3.2 Plant Not in Model Set

Next, in [2], the case is considered when the plant (and noise covariances) is not in the model set. Specifically, let the true plant be given by

$$P_o(z) = P(z, \alpha_o)[1 + \Delta(z, \alpha_o)] \tag{14}$$

for some $\alpha_o \in \mathcal{A}$, where now none of the Kalman filters is optimal. The question is: if one acts as if the true plant were in the model set, and the probabilities are computed as before, then what actually happens?

The answer is that the quantities $p(\alpha_i|z^t)$ computed from (5), which might now be more correctly referred to as *pseudo-probabilities*, behave exactly as before, namely, that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \tag{15}$$

where

$$i_* = \arg \min_{i \in [1, \ldots, N]} V_i \tag{16}$$

Hence, the algorithm selects the plant model $P(z, \alpha_{i_*})$ in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$ which is closest to the true plant $P_o(z)$ in the sense of minimizing V_i over all i. As shown in [2], this is equvalent to minimizing the *Kullback information measure* computed on an asymptotic per sample basis.

We remark also that if $\Phi_i(\omega)$ and $\Phi_o(\omega)$ are the model and plant output spectrum, respectively, then

$$\Phi_i(\omega) = W_i(e^{j\omega})\Omega_i W_i^T(e^{-j\omega}) \tag{17}$$

$$\Phi_o(\omega) = W_o(e^{j\omega})\Omega_o W_o^T(e^{-j\omega}) \tag{18}$$

where $W_i^{-1}(z)$ and $W_o^{-1}(z)$ are the transfer functions from the measurement sequence $y(t)$ to the model innovations $\epsilon_i(t)$ and true innovations $\epsilon_o(t)$, respectively. Note that these transfer functions can always be selected to be stable and stably invertible. Hence, we can compute Σ_i by

$$\Sigma_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T \, d\omega \tag{19}$$

and also

$$\mathrm{tr}(\Omega_i^{-1}\Sigma_i) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)]\, d\omega \tag{20}$$

Finally, we get

$$V_i = \ln\det(\Omega_i) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)]\, d\omega \tag{21}$$

4 The Effect of Feedback and Exogenous Inputs

We work now to consider the arrangement depicted in Figure 3. Let the plant system be described by

$$y((t) = P_o(z)u(t) + W_o(z)\nu_o(t) \tag{22}$$

where $\nu_o(t)$ is a zero mean white noise sequence with covariance matrix Ω_o. The feedback system is

$$u(t) = C(z)[r(t) - y(t)] \tag{23}$$

where $r(t)$ is a stationary zero mean sequence with spectrum $\Phi_r(\omega)$. The controller $C(z)$ may be one of the avaiable set of controllers $C_1(z), \ldots, C_N(z)$. Recall that each controller $C_j(z)$ is designed to stabilize any plant in the set $\mathcal{P}_j$ defined in (2). Observe that if the true plant $P_o(z)$ is not in this set, then the closed loop system may be unstable if $C_j(z)$ is applied.

The block labeled KF_i denotes a Kalman filter which is designed for the plant model

$$y(t) = P(z,\alpha_i)u(t) + W(z,\alpha_i)\nu_i(t) \tag{24}$$

where $\nu_i(t)$ is a zero mean white noise sequence with covariance matrix Ω_i. Hence, the innovations sequence $\varepsilon_i(t)$ from KF_i is given by

$$\varepsilon_i(t) = W^{-1}(z,\alpha_i)[y(t) - P(z,\alpha_i)u(t)] \tag{25}$$

The difference here in relation to [2] is: (i) the inclusion of an external input $u(t)$ to the plant and the Kalman filter, and (ii) the generation of that input by a combination of an exogenous input $r(t)$ and a feedback compensator $C(z)$. We now examine how the filter selection algorithm (5) behaves in two cases, namely, when $C(z)$ stabilizes $P_o(z)$ and when it does not.

4.1 Stabilizing Feedback

Suppose that the feedback compensator $C(z)$ stabilizes $P_o(z)$. In this case the pseudo-probabilities $p(\alpha_i|z^t)$ are again computed from (5), but now we take the set of measurements as $z^t = \{y(k), u(k) : k = 1, \ldots, t\}$. The result (15), (16) is the same as before: the algorithm selects the plant model that is closest in the sense of the information measure, i.e., that plant

model $P(z,\alpha_i), W(z,\alpha_i)$ which is closest to the true plant $P_o(z), W_o(z)$ in the sense of minimizing V_i in (21). What is different is the formula for V_i. To compute this first observe that (we drop the explicitly shown dependence on z, t, and α_i to simplify notation)

$$\varepsilon_i = W_i^{-1}[(P_o - P_i)u + W_o\nu_o] \tag{26}$$

and under the feedback action

$$u = C(I + P_oC)^{-1}[r - W_o\nu_o] \tag{27}$$

Note that by saying that $C(z)$ stabilizes $P_o(z)$ we mean that the transfer functions $C(I + P_oC)^{-1}$, $(I + P_oC)^{-1}$, and $P_o(I + P_oC)^{-1}$ are all stable. Hence, we get

$$\varepsilon_i = H_iW_o\nu_o + G_ir \tag{28}$$

where

$$H_i = W_i^{-1}(I + P_iC)(I + P_oC)^{-1} \tag{29}$$

and

$$G_i = W_i^{-1}(P_o - P_i)C(I + P_oC)^{-1} \tag{30}$$

Assuming that $r(t)$ and $\nu_o(t)$ are uncorrelated, then

$$\Phi_{\varepsilon_i}(\omega) = H_i(e^{j\omega})\Phi_o(\omega)H_i^T(e^{-j\omega}) + G_i(e^{j\omega})\Phi_r(\omega)G_i^T(e^{-j\omega}) \tag{31}$$

and hence

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\varepsilon_i}(\omega)\, d\omega \tag{32}$$

To get better insight into this formula, observe what happens when we model the plant input-output dynamics correctly, that is, when $P_i = P_o$. In this case the presence of the external input has no effect and we get

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T\, d\omega \tag{33}$$

as before. In the more general case with a stabilizing feedback, the above formula for Σ_i is effected by errors in both the dynamical model due to $P_o(z) - P(z,\alpha_i)$ and in the noise model via $W_o(z) - W(z,\alpha_i)$. Signal to noise ratio is also relevant, for example, if $\Phi_r(\omega) >> \Phi_o(\omega)$, then the term involving $G_i(z)$ tends to dominate, i.e., the noise model error is not as important as the dynamical model error.

Other insights can also be obtained by using the expressions derived in [5], such as

$$\varepsilon_i = \nu_o + W_i^{-1}\tilde{T}_i \begin{pmatrix} u \\ \nu_o \end{pmatrix} \tag{34}$$

where

$$\tilde{T}_i = [P_o - P_i \quad W_o - W_i] \tag{35}$$

Further interpretations will be provided in the full paper.

4.2 Unstabilizing Feedback

It could be that the adaptive selection algorithm (5) picks out a control which would stabilize one of the plant models, but de-stabilizes the true plant. The question arises as to what happens if the same algorithm is used to drive the adaptive switch. Clearly, the limiting process in (21) will no longer be valid.

It is shown in the full paper that if the dominnent unstable mode of the closed loop system is at $z = \zeta$, $|\zeta| > 1$, then the selection algorithm identifies that plant $P(z, \alpha_i)$ for which $P(\zeta, \alpha_i)$ is "most like" $P_o(\zeta)$. If $P(\zeta, \alpha_i) = P_o(\zeta)$, then the controller $C_i(z)$ appropriate for $P(z, \alpha_i)$, when connected to $P_o(z)$, will not produce an unstable mode at ζ. Nor will it do so if $P(\zeta, \alpha_i) \approx P_o(\zeta)$, because the controller is robustly designed to stabilize all plants in the set $\mathcal{P}_i$ defined in (2). However, there is no a priori guaranty that it will give no other unstable mode $\zeta^+ \neq \zeta$. So despite the fact that there is a systematic way of switching out an unstable controller, there is no guaranty that instability can be instantaneously repaired. One could randomly select any alternative control from the set $C_1(z), \ldots, C_N(z)$, or better, whenever instability is deemed to occur, say whenever $\|y(t)\|$ exceeds some threshhold, a stabilizing, but low authority controller, can be switched in until the adaptive system selects a new plant/ controller.

5 Algorithm Modifications

The pseudo-probability algorithm (5) can be modified in a number of ways. The influence of past data, which tends to make the algorithm sluggish, can be reduced by discarding old data. For example, suppose that the controller selection is to be based on the smallest

$$V_i(t) = \ln \det(\Omega_i) + \frac{1}{t} \sum_{k=1}^{t} \lambda^{t-k} \frac{1}{2} \varepsilon_i^T(k) \Omega_i^{-1} \varepsilon_i(k) \tag{36}$$

where $\lambda \in (0, 1]$ is the usual "forgetting factor" which exponentially discards old data. Let $p_i(t)$ denote the pseudo-probability associated with $V_i(t)$ above. The algorithm (5) now becomes

$$p_i(t) = \frac{[p_i(t-1)]^\lambda \sqrt{\det(\Omega_i^{-1})} \exp\{-\frac{1}{2}\varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)\}}{\sum_{j=1}^{N} [p_j(t-1)]^\lambda \sqrt{\det(\Omega_j^{-1})} \exp\{-\frac{1}{2}\varepsilon_j^T(t)\Omega_j^{-1}\varepsilon_j(t)\}} \tag{37}$$

Following the derivations in [2], we now get

$$\ln L_i(t) = \lambda \ln L_i(t-1) + \frac{1}{2}[\varepsilon_{i_*}^T(t)\Omega_{i_*}^{-1}\varepsilon_{i_*}(t) - \varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)] \tag{38}$$

where

$$i_* = \arg\min_i V_i(t) \tag{39}$$

and for $i \neq i_*$,

$$L_i(t) = \frac{p_i(t)}{p_{i_*}(t)} \tag{40}$$

As in the previous cases, it can be shown that as $t \to \infty$,

$$p_i(t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \tag{41}$$

exponentially fast.

Other variants will be explored in the full paper.

6 Adaptive Selection

In the full paper we will provide a detailed analysis when the the control is given by

$$u(t) = \sum_{i=1}^{N} \beta_i(t) u_i(t) \tag{42}$$

with

$$u_i(t) = C_i(z)[r(t) - y(t)], \; i = 1, \ldots, N \tag{43}$$

where $\beta_1(t), \ldots, \beta_N(t)$ are selection coefficients, or weights, which are determined from the pseudo- probabilities $p_1(t), \ldots, p_N(t)$. We will examine two types of adaptive selection mechanisms, namely (i) abrupt selection, and (ii) smooth selection. In either case it may be necessary to also provide for a back-up controller

$$u_0(t) = C_0(z)[r(t) - y(t)] \tag{44}$$

which is of low-authority, but stabilizes all plants in $\mathcal{P}$. This control can be switched in whenever an "instability" is deemed to occur.

6.1 Abrupt Adaptation

In this case we adjust the weights $\beta_i(t)$ abruptly. For example:

$$\beta_i(t) = \begin{cases} 1, & p_i(t-1) < p_{th} \leq p_i(t) \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

where $p_{th} \in (0,1)$ is some threshhold value, probably close to one. Hence , the current control is held until one of the pseudo-probabilities is sufficiently large, and then a new control is switched on in its place.

6.2 Smooth Adaptation

To avoid unpleasentries associated with abrupt switching one can swithch smoothly. The most obvious choice is simply to set

$$\beta_i(t) = p_i(t) \tag{46}$$

Adaptive Control via Finite Modeling and Robust Control

Robert L. Kosut* and Brian D.O. Anderson†

Abstract: This papers examines an adaptive control scheme which involves the selection of a fixed controller out of a finite collection, where each controller is robustly designed to account for the true plant being in a not necessarily small set of uncertainty. There are no adaptive "parameters" in the conventional sense where adjustments are made to either model or controller parameters. The parameters here involve pseudo-probabilities or weights assigned to each of the finite controllers. These quantities are computed recursively from the measured data.

1 Introduction

Our concern is with the adaptive control of plants with both parametric (structured) uncertainty and dynamical (unstructured) uncertainty. The plant is linear, and subject to disturbance and measurement noise. Knowing the value of the uncertain parameters allows for the design of a robust controller to cope with the unstructured uncertainty. Moreover, there is no robust controller, i.e., one that is linear-time-invariant, that will handle the entire spread of structured and unstructured uncertainty. Hence, there is the need for an adaptive control.

More precisely, let $\mathcal{P}$ denote the family of plants given by

$$\begin{aligned}\mathcal{P} \;=\; & \{P(z) = P(z,\alpha)[1+\Delta(z,\alpha)] : \alpha \in \mathcal{A}, \\ & |\Delta(e^{j\omega},\alpha)| \le |\delta(e^{j\omega},\alpha)|, \forall \omega \in [-\pi,\pi]\}\end{aligned} \tag{1}$$

where α is the structured parameter constrained to a subset $\mathcal{A}$ of $\mathbf{R}^p$, and $\Delta(z,\alpha)$ is the unstructured uncertainty bounded by the weighting function $\delta(z,\alpha)$. Assume that there is a finite number of parameter "design" values $\alpha_1,\ldots,\alpha_N$, each in $\mathcal{A}$, such that for *any* $\alpha \in \mathcal{A}$, not necessarily equal to $\alpha_1,\ldots,\alpha_N$, there exists a controller $C_i(z)$, tuned to α_i for some i, which acheives satisfactory performance in the face of the unstructured uncertainty, and in the face of the difference between the actual value of α and the design value α_i. For example, the i-th controller $C_i(z)$ in the set $C_1(z),\ldots,C_N(z)$, provides robust performance for any plant in the family

$$\mathcal{P}_i = \{P(z) \in \mathcal{P} : \|\alpha - \alpha_i\| \le m_i\} \tag{2}$$

where $\mathcal{P}_1,\ldots,\mathcal{P}_N$ completely cover the plant set $\mathcal{P}$, that is

$$\mathcal{P} \subset \bigcup_{i=1}^{N} \mathcal{P}_i \tag{3}$$

Our task in this paper is to explain how, when α is unknown or slowly varying, the controller $C_i(z)$ can be selected to control the plant. This type of adaptive controller is not like the conventional ones where either model or controller parameters are directly adjusted. Here the parameters are contained in the mechanism for switching amongst the preselected robust controllers, which is essentially a gain scheduling proceedure, but is adaptive in the sense that the schedule is being learned from the measured data. The gain schedule is usually set in advance, for example, in a flight control system the gain schedule is a predetermined function of the Mach number and aerodynamic pressure. One of the interesting possible advantages of this method is that although the plant may have a large number of uncertain parameters (α in the above notation), it is possible that only a few controllers are required, and hence only a few parameters in the selection mechanism. Also the individual robust controllers can be based on uncertainty in physical parameters rather than canonical parameters, such as transfer function coefficients, as used in the conventional parameter estimation techniques, e.g., least squares with a linear regression model.

The idea of using preselected robust controllers as described above is due to K. Poola [1], who also proposed a particular smooth adaptive selection algorithm. Here, we examine the adaptive selection mechanism as discussed in [2] and [3], which will be described in detail in the sequel. In [3] this scheme is referred to as a Multiple Model Adaptive Control (MMAC), but the individual controllers are not

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necessarily selected to be robust in the manner described above. Here we also provide an analysis of the convergence properties of the adaptive selection algorithm following the analysis in [2].

2 Structure of the Adaptive Controller

Figure 1 depicts the general adaptive set-up that we consider. The design of each controller $C_i(z)$ is not an adaptive control design, but rather, a robust control design task involving the plant family $\mathcal{P}$ and the coverings $\mathcal{P}_i$. The signals used to adaptively select the controller are the innovations sequences, or prediction errors, denoted by $\varepsilon_1(t), \ldots, \varepsilon_N(t)$, where each is obtained from the Kalman filters denoted $KF_1, \ldots, KF_N$.

Although there are many structures for the individual controllers $C_i(z)$, one structure which utilizes the prediction errors is shown in Figure 2. In this case $C_i(z)$ consists of an observer based state feedback controller together with an auxiliary signal obtained by processing the observer innovations sequence $\varepsilon_i(t)$ through a stable transfer function $Q_i(z)$. An important aspect regarding the flexibility of this controller structure is that each $C_i(z)$ is a parametrization, in terms of stable $Q_i(z)$, of all stabilizing controllers of the plant model $P(z, \alpha_i)$, see, e.g.,[4]. That is, *all* controllers which stabilize $P(z, \alpha_i)$ are obtained by letting $Q_i(z)$, referred to as the *Youla parameter*, range over all stable transfer functions.

The results in [2] are first reviewd. We then provide the extension to the feedback case. The adaptation mechanism is the real object of interest in this paper. We set it up using the ideas of [2] and [3]. In broad outline, the idea is this: Design predictors (e.g., Kalman filters) which are optimal for each of the plants $P(z, \alpha_i)$, $i = 1, \ldots, N$. (Their design also requires values to be assigned to noise covariances.) We then process the innovations, or prediction errors, from each predictor and compute certain quantities, roughly, the scalar sample covariance. The index value $i_* \in [1, \ldots, N]$ whose predictor produces the smallest sample covariance is selected and the corresponding $C_{i_*}(z)$ controller is used to control the plant.

3 Review of [2]

Two issues are examined in [2]. First, when the plant is in the model set, and secondly, when it is not.

3.1 Plant in Model Set

Let a plant be drawn from the collection $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, say the plant is $P(z, \alpha_{i_o})$. Suppose the plant has disturbances and measurement noise, but no exogenous input. Let N Kalman filters, each tuned to $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, be connected to the plant. Let the *asymptotic* design innovation covariances be Ω_i and let $\varepsilon_i(t)$ denote the sequence obtained from the i-th filter connected to $P(z, \alpha_{i_o})$, at the point in the filter where the innovation sequence would normally be observed. Recall that the quantities $\varepsilon_i(t)$ and Ω_i are available from the Kalman filter and covariance equations, respectively. Moreover, the design covariance Ω_i is computible in adavance of the measurements. Note also that $\varepsilon_{i_o}(t)$ is actually an innovation sequence, but $\varepsilon_i(t)$ for $i \neq i_o$ is not in general. Define, for $i \neq i_o$,

$$L_i(t) = \frac{p(\alpha_i|z^t)}{p(\alpha_{i_o}|z^t)} \tag{4}$$

where z^t denotes the sequence of measurements $y(1), \ldots, y(t)$, and $p(\alpha|z^t)$ are the *a posteriori probabilities* which, assuming gaussian distributions, are sequentially computible from

$$\begin{aligned} p(\alpha_i|z^t) &= \frac{p(\alpha_i|z^{t-1})\sqrt{\det(\Omega_i^{-1})}E_i(t)}{\sum_{j=1}^N p(\alpha_j|z^{t-1})\sqrt{\det(\Omega_j^{-1})}E_j(t)} \\ E_i(t) &= \exp\{-\frac{1}{2}\varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)\} \end{aligned} \tag{5}$$

It follows immediately that

$$\begin{aligned} \ln L_i(t) &= \ln L_i(t-1) \\ &\quad + \frac{1}{2}[\varepsilon_{i_o}^T(t)\Omega_{i_o}^{-1}\varepsilon_{i_o}(t) - \varepsilon_i^T(t)\Omega_i^{-1}\varepsilon_i(t)] \\ &\quad + \frac{t}{2}\ln\frac{\det(\Omega_{i_o})}{\det(\Omega_i)} \end{aligned} \tag{6}$$

Thus, as $t \to \infty$, and for all $i \neq i_o$,

$$\frac{2}{t}\ln[L_i(t)] \to -(V_i - V_{i_o}) \tag{7}$$

where for *all* i,

$$V_i = \ln\det(\Omega_i) + \mathrm{tr}(\Omega_i^{-1}\Sigma_i) \tag{8}$$

with

$$\Sigma_i = \lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^t \varepsilon_i(k)\varepsilon_i^T(k) \tag{9}$$

$$= \mathcal{E}\{\varepsilon_i(t)\varepsilon_i^T(t)\} \tag{10}$$

The expectation operator $\mathcal{E}(\cdot)$ is of course taken with respect to the noise processes. By definition of optimality,

$$V_{i_o} < V_i, \ \forall i \neq i_o \tag{11}$$

and thus, $\forall i \neq i_o$, as $t \to \infty$,

$$\frac{p(\alpha_i|z^t)}{p(\alpha_{i_o}|z^t)} \to 0 \tag{12}$$

exponentially fast with asymptotic convergence rate $\exp\{-(V_i - V_{i_o})t\}$. This means that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_o \\ 0, & i \neq i_o \end{cases} \tag{13}$$

and hence, the correct plant model is captured. Intuitively this is as expected, because the plant $P(z, \alpha_{i_o})$ is in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, and moreover, one of the Kalman filters is optimal.

3.2 Plant Not in Model Set

Next, in [2], the case is considered when the plant (and noise covariances) is not in the model set. Specifically, let the true plant be given by

$$P_o(z) = P(z, \alpha_o)[1 + \Delta(z, \alpha_o)] \tag{14}$$

for some $\alpha_o \in \mathcal{A}$, where now none of the Kalman filters is optimal. The question is: if one acts as if the true plant were in the model set, and the probabilities are computed as before, then what actually happens?

The answer is that the quantities $p(\alpha_i|z^t)$ computed from (5), which might now be more correctly referred to as *pseudo-probabilities*, behave exactly as before, namely, that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \tag{15}$$

where

$$i_* = \arg \min_{i \in [1, \ldots, N]} V_i \tag{16}$$

Hence, the algorithm selects the plant model $P(z, \alpha_{i_*})$ in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$ which is closest to the true plant $P_o(z)$ in the sense of minimizing V_i over all i. As shown in [2], this is equvalent to minimizing the *Kullback information measure* computed on an asymptotic per sample basis.

We remark also that if $\Phi_i(\omega)$ and $\Phi_o(\omega)$ are the model and plant output spectrum, respectively, then

$$\Phi_i(\omega) = W_i(e^{j\omega})\Omega_i W_i^T(e^{-j\omega}) \tag{17}$$

$$\Phi_o(\omega) = W_o(e^{j\omega})\Omega_o W_o^T(e^{-j\omega}) \tag{18}$$

where $W_i^{-1}(z)$ and $W_o^{-1}(z)$ are the transfer functions from the measurement sequence $y(t)$ to the model innovations $\varepsilon_i(t)$ and true innovations $\varepsilon_o(t)$, respectively. Note that these transfer functions can always be selected to be stable and stably invertible. Hence, we can compute Σ_i by

$$\Sigma_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T \, d\omega \tag{19}$$

and also

$$\mathrm{tr}(\Omega_i^{-1}\Sigma_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)] \, d\omega \tag{20}$$

Finally, we get

$$V_i = \ln \det(\Omega_i) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)] \, d\omega \tag{21}$$

4 The Effect of Feedback and Exogenous Inputs

We work now to consider the arrangement depicted in Figure 3. Let the plant system be described by

$$y(t) = P_o(z)u(t) + W_o(z)\nu_o(t) \tag{22}$$

where $\nu_o(t)$ is a zero mean white noise sequence with covariance matrix Ω_o. The feedback system is

$$u(t) = C(z)[r(t) - y(t)] \tag{23}$$

where $r(t)$ is a stationary zero mean sequence with spectrum $\Phi_r(\omega)$. The controller $C(z)$ may be one of the available set of controllers $C_1(z), \ldots, C_N(z)$. Recall that each controller $C_j(z)$ is designed to stabilize any plant in the set $\mathcal{P}_j$ defined in (2). Observe that if the true plant $P_o(z)$ is not in this set, then the closed loop system may be unstable if $C_j(z)$ is applied.

The block labeled KF_i denotes a Kalman filter which is designed for the plant model

$$y(t) = P(z, \alpha_i)u(t) + W(z, \alpha_i)\nu_i(t) \tag{24}$$

where $\nu_i(t)$ is a zero mean white noise sequence with covariance matrix Ω_i. Hence, the innovations sequence $\varepsilon_i(t)$ from KF_i is given by

$$\varepsilon_i(t) = W^{-1}(z, \alpha_i)[y(t) - P(z, \alpha_i)u(t)] \tag{25}$$

The difference here in relation to [2] is: (i) the inclusion of an external input $u(t)$ to the plant and the Kalman filter, and (ii) the generation of that input by a combination of an exogenous input $r(t)$ and a feedback compensator $C(z)$. We now examine how the filter selection algorithm (5) behaves in two cases, namely, when $C(z)$ stabilizes $P_o(z)$ and when it does not.

4.1 Stabilizing Feedback

Suppose that the feedback compensator $C(z)$ stabilizes $P_o(z)$. In this case the pseudo-probabilities $p(\alpha_i|z^t)$ are again computed from (5), but now we take the set of measurements as $z^t = \{y(k), u(k) : k = 1, \ldots, t\}$. The result (15), (16) is the same as

before: the algorithm selects the plant model that is closest in the sense of the information measure, i.e., that plant model $P(z,\alpha_i), W(z,\alpha_i)$ which is closest to the true plant $P_o(z), W_o(z)$ in the sense of minimizing V_i in (21). What is different is the formula for V_i. To compute this first observe that (we drop the explicitly shown dependence on z, t, and α_i to simplify notation)

$$\varepsilon_i = W_i^{-1}[(P_o - P_i)u + W_o\nu_o] \tag{26}$$

and under the feedback action

$$u = C(I + P_oC)^{-1}[r - W_o\nu_o] \tag{27}$$

Note that by saying that $C(z)$ stabilizes $P_o(z)$ we mean that the transfer functions $C(I+P_oC)^{-1}$, $(I+P_oC)^{-1}$, and $P_o(I+P_oC)^{-1}$ are all stable. Hence, we get

$$\varepsilon_i = H_iW_o\nu_o + G_ir \tag{28}$$

where

$$H_i = W_i^{-1}(I + P_iC)(I + P_oC)^{-1} \tag{29}$$

and

$$G_i = W_i^{-1}(P_o - P_i)C(I + P_oC)^{-1} \tag{30}$$

Assuming that $r(t)$ and $\nu_o(t)$ are uncorrelated, then

$$\begin{aligned}\Phi_{\varepsilon_i}(\omega) &= H_i(e^{j\omega})\Phi_o(\omega)H_i^T(e^{-j\omega}) \\ &\quad + G_i(e^{j\omega})\Phi_r(\omega)G_i^T(e^{-j\omega})\end{aligned} \tag{31}$$

and hence

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{\varepsilon_i}(\omega)\,d\omega \tag{32}$$

To get better insight into this formula, observe what happens when we model the plant input-output dynamics correctly, that is, when $P_i = P_o$. In this case the presence of the external input has no effect and we get

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T\,d\omega \tag{33}$$

as before. In the more general case with a stabilizing feedback, the above formula for Σ_i is effected by errors in both the dynamical model $P_o(z) - P(z,\alpha_i)$ and in the noise model via $W_o(z) - W(z,\alpha_i)$. Signal to noise ratio is also relevant. For example, if $\Phi_r(\omega) >> \Phi_o(\omega)$, then the term involving $G_i(z)$ tends to dominate, i.e., the noise model error is not as important as the dynamical model error.

Other insights can also be obtained by using the expressions derived in [5], such as

$$\varepsilon_i = \nu_o + W_i^{-1}\tilde{T}_i\begin{pmatrix} u \\ \nu_o \end{pmatrix} \tag{34}$$

where

$$\tilde{T}_i = [P_o - P_i \quad W_o - W_i] \tag{35}$$

4.2 Unstabilizing Feedback

It could be that the adaptive selection algorithm (5) picks out a control which would stabilize one of the plant models, but de-stabilizes the true plant. The question arises as to what happens if the same algorithm is used to drive the adaptive switch. Clearly, the limiting process in (21) will no longer be valid.

Roughly what happens is this: if the dominant unstable mode of the closed loop system is at $z = \zeta$, $|\zeta| > 1$, then the selection algorithm identifies that plant $P(z,\alpha_i)$ for which $P(\zeta,\alpha_i)$ is "most like" $P_o(\zeta)$. If $P(\zeta,\alpha_i) = P_o(\zeta)$, then the controller $C_i(z)$ appropriate for $P(z,\alpha_i)$, when connected to $P_o(z)$, will not produce an unstable mode at ζ. Nor will it do so if $P(\zeta,\alpha_i) \approx P_o(\zeta)$, because the controller is robustly designed to stabilize all plants in the set $\mathcal{P}_i$ defined in (2). However, there is no a priori guaranty that it will give no other unstable mode $\zeta^+ \neq \zeta$. So despite the fact that there is a systematic way of switching out an unstable controller, there is no guaranty that instability can be instantaneously repaired. One could randomly select any alternative control from the set $C_1(z),\ldots,C_N(z)$, or better, whenever instability is deemed to occur, say whenever $\|y(t)\|$ exceeds some threshhold, a stabilizing, but low authority controller, can be switched in until the adaptive system selects a new plant/ controller.

5 Algorithm Modifications

The pseudo-probability algorithm (5) can be modified in a number of ways. The influence of past data, which tends to make the algorithm sluggish, can be reduced by discarding old data. For example, suppose that the controller selection is to be based on the smallest

$$V_i(t) = \ln\det(\Omega_i) + \frac{1}{t}\sum_{k=1}^{t}\lambda^{t-k}\frac{1}{2}\varepsilon_i^T(k)\Omega_i^{-1}\varepsilon_i(k) \tag{36}$$

where $\lambda \in (0,1]$ is the usual "forgetting factor" which exponentially discards old data. Let $p_i(t)$ denote the pseudo-probability associated with $V_i(t)$ above. The algorithm (5) now becomes

$$p_i(t) = \frac{[p_i(t-1)]^\lambda\sqrt{\det(\Omega_i^{-1})}E_i(t)}{\sum_{j=1}^{N}[p_j(t-1)]^\lambda\sqrt{\det(\Omega_j^{-1})}E_j(t)} \tag{37}$$

Define,

$$i_* = \arg\min_i V_i(t) \tag{38}$$

and for $i \neq i_*$,

$$L_i(t) = \frac{p_i(t)}{p_{i_*}(t)} \tag{39}$$

As in the previous cases, it can be shown that as $t \to \infty$,

$$p_i(t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \qquad (40)$$

exponentially fast.

6 Concluding Remarks

We have revisited some earlier work on multiple model adaptive control and have introduced some of the ideas of robust control into the procedure. The controller parametrization in Figure 2 leads one to hope that a more robust multiple model controller has the form

$$u(t) = \sum_{i=1}^{N} p_i(t) \{Q_i(z)\varepsilon_i(t)\} \qquad (41)$$

Compare this to the "natural" choice given by

$$u(t) = \sum_{i=1}^{N} p_i(t) \{C_i(z)[r(t) - y(t)]\} \qquad (42)$$

Apparently a periodic controller switching mechanism can lead to the above "robust" form, e.g., [1], but this is not verified for the simpler switching mechanism considered here. In any event, it may be necessary to also provide for a back-up controller

$$u_0(t) = C_0(z)[r(t) - y(t)] \qquad (43)$$

which is of low-authority, but stabilizes all plants in $\mathcal{P}$. This control can be switched in whenever an "instability" is deemed to occur.

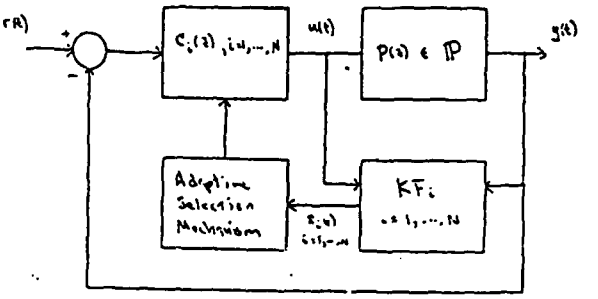

Figure 1: Adaptive Control Structure

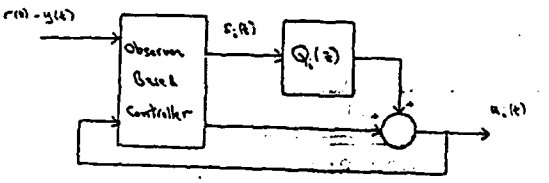

Figure 2: i-th Controller Structure

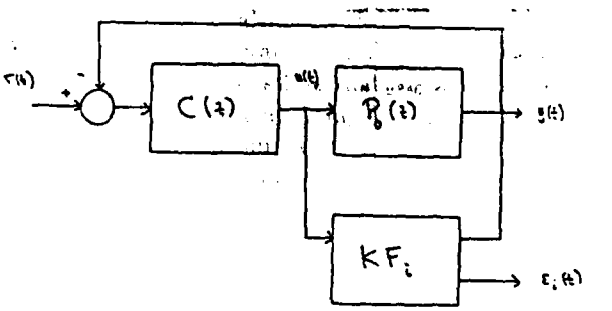

Figure 3: Fixed Feedback and Multiple Filters

References

[1] K. Poola, Personal communication and unscheduled presentation at the MTNS, June 1987, Phoenix, AZ.

[2] B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Prentice-Hall, Englewood Cliffs, NJ, 1979.

[3] M. Athans, D. Castanon, K-P Dunn, C.S. Greene, W.H. Lee, N.R. Sandell, Jr., and A.S. Wilsky, "The stochastic control of the F-8C aircraft using a multiple-model-adaptive-control (MMAC) method-Part 1: equilibrium flight", *IEEE Trans. on Aut. Contr.*, vol AC-22, no 5, Oct. 1977, pp 768-780.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1983.

[5] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

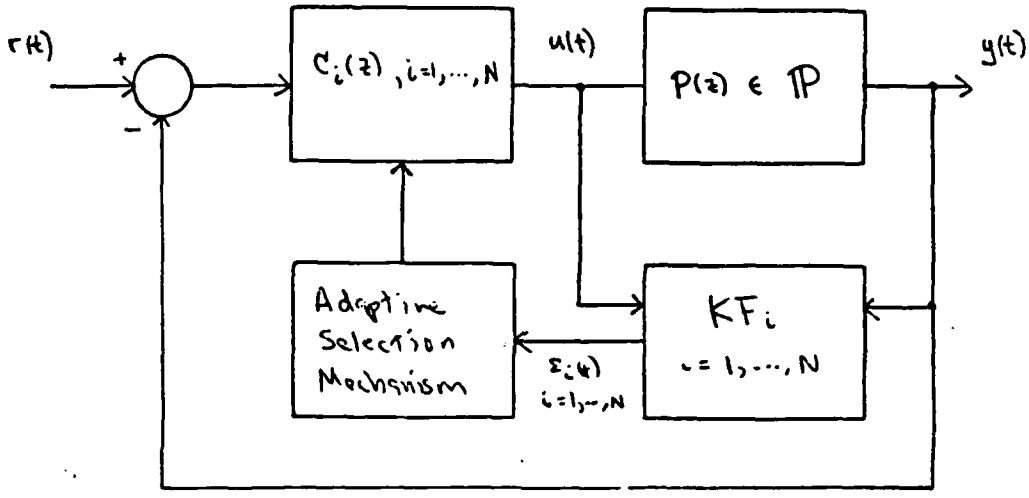

Figure 1: Adaptive Control Structure

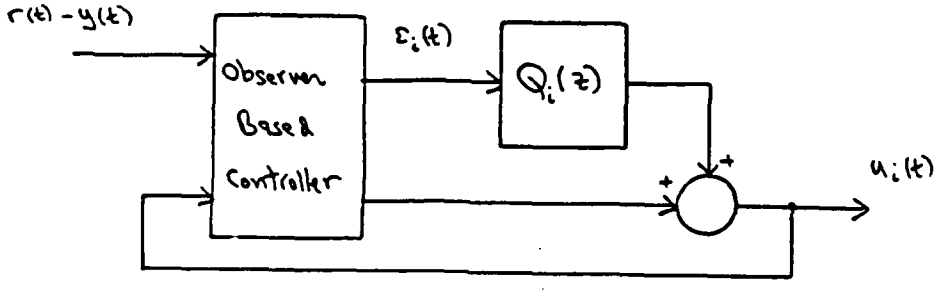

Figure 2: i-th Controller Stucture

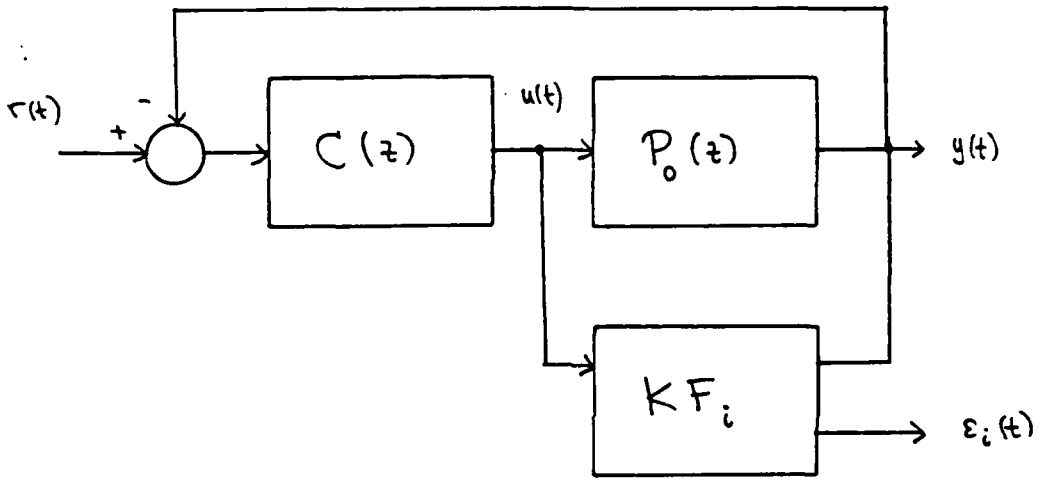

Figure 3: Fixed Feedback and Multiple Filters

On The Use of The Method of Averaging For The Stability Analysis of Adaptive Linear Control Systems

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Abstract - *The uses and limitations of the method of averaging are discussed with application to linear parameter adaptive systems. The method of averaging has shown to provide a means to explicitly determine some of the important aspects of adaptive system performance, but there are inherent limitations, e.g., slow adaptation and parameters restricted to trajectories within the constant parameter stability set. Remedies to alleviate these restrictions are discussed, based primarily on the use of a fixed point analysis.*

I. INTRODUCTION

Although uncertainty underlies the reason for using an adaptive control system, too much of the wrong kind of uncertainty may be cause for grief. For example, simulations of some simple adaptive control systems, under apparently minor non-ideal conditions, have shown degraded performance and even instabilities, e.g., Rohrs et al.(1981, 1982). In this paper we examine the uses and limitations of the method of averaging for the stability analysis of adaptive linear control systems.

A mathematical treatment of averaging can be found in Chapter 5 of Hale (1969) or in Chapter 4 of Guckenheimer and Holmes (1983). Applications to adaptive systems can be found, to name a few sources, in: Ljung and Soderstrom (1983), where stochastic averaging methods are employed to study the asymptotic parameter trajectories of recursive parameter estimation algorithms in system identification methods; Astrom (1983, 1984) showing how the method of averaging explains instabilities and drift; Reidle and Kokotovic (1985) study averaging for the linearized adaptive system and establish a sharp stability-instability frequency domain test; Kosut, Anderson, and Mareels (1987) on the relation between averaging and persistent excitation; Bodson et al.(1986) on nonlinear averaging analysis and determining the rate of convergence; Riedle and Kokotovic (1986) on the slow integral manifold analysis; and the monograph by Anderson et al.(1986), which uses the method of averaging and a total stability approach.

Uses and Limitations

The method of averaging provides a great deal of insight into the behavior of adaptive systems. The analysis provides quantitative tests for determining local stability and instability, rate of convergence, and robustness to unmodeled effects. But, it is not a panacea, as there are inherent restrictions in the analysis. In the first place, the speed of adaptation required to satisfy the *theoretical* conditions is most often far below that as determined from simulations. Secondly, the results are valid only when the parameters are restricted to a subset of the constant parameter stability set. Projection techniques can be employed to restrict the parameters as required, but in many practical cases this is not feasible nor even necessary, e.g., in output error system identification and adaptive control, momentary unstable parameter settings can induce very rapid learning, e.g., Anderson (1985). However, the exact mechanism is not well understood. Also, slow adaptation, if it is used, can be counter-productive in some instances because performance can be below par for the long period of time it takes for the parameters to adjust.

Transient Analysis

To remove these restrictions requires understanding the *transient behavior* of adaptive systems. Some preliminary results are reported in Kosut et al.(1987) and Kosut and Bitmead (1986). The tools for analysis involve a combination of small gain theory, passivity, and the method of averaging, with these all linked together by the Contraction Mapping Principal. Some of these ideas will be briefly described in this paper.

Beyond Hand Calculations

Although each of these tools, in principal, involves straightforward calculations, even simple examples can just barely be worked out by hand. It is clear that the level of complexity of a realistic adaptive system is well beyond hand calculation. Hence, in order for any of the above mentioned analytic methods to be of practical benefit, it is imperitve to develop "user-friendly" software tools which provide the requisite nonlinear analysis.

At the present time, aside from simulation capability, there are no available software tools for dealing principally with adaptive systems. This is a research issue in both mathematics and computation, and it is one that is essen-

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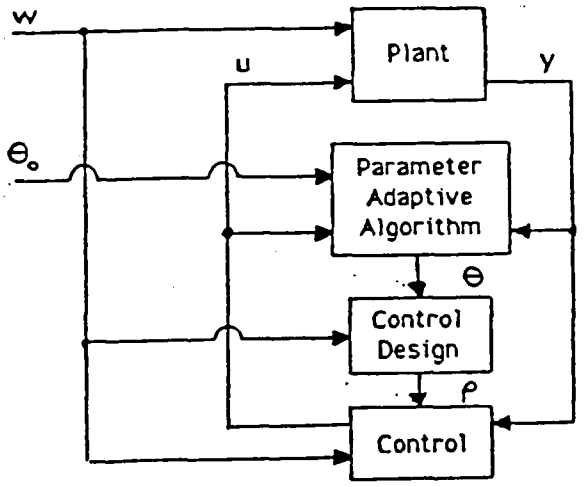

Figure 1: Parameter adaptive control system

tial to a continuing study of adaptive systems. This point will not be pursued here, but it is sometimes an overlooked research issue, and thus warrents more than a passing remark.

II. LINEAR ADAPTIVE SYSTEMS

Adaptive System Structure

Figure 1 depicts a parameter adaptive control system where $u(t)$ and $y(t)$ are the vectors of measured inputs and outputs, $w(t)$ is a vector of exogenous inputs, i.e., references, disturbances, and noise sources, and the vector of adaptive parameters is $\theta(t)$ with initial value θ_o. The adaptive parameter vector is mapped into a "control" parameter vector $\rho(t)$ by some design rule, typically an implicitly defined memoryless nonlinear function . For example, in control system synthesis, one can use any number of model (parameter) based methods, see e.g., Safonov et al. (1981) on LQG based designs or Vidyasagar (1985) on the stable factorization approach. In system identification, on the other hand, we normally have a the simple design rule $\theta = \rho$. A more extensive display of how the various standard adaptive identification and control systems fit into the structure of Figure 1 is provided in Anderson et al.(1986).

In order to illustrate the basic ideas, we will assume here that the parameter adaptive algorithm is contiuously adjusted. In practice, parameter adjustments would be either at discrete times or the control signal is the output of a digital computer.

We will also assume that for fixed values of the control parameter vector ρ, the plant and controller are both linear-time-invariant with rational proper transfer functions. Thus, the parameter adaptive system of Figure 1 is an *adaptive linear control system*, which can be described by the coupled set of ordinary differential equations

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)w(t) \\ \dot{\theta} &= \gamma q(t, x, \theta) \end{aligned} \tag{1}$$

where $\theta(t) \in \mathbf{R}^p$ is the adaptive parameter, and $x(t) \in \mathbf{R}^n$ is the system state, consisting of plant, controller, filters, and possibly, reference model states. The matrix functions $A(\theta), B(\theta)$ are determined by the design rule $\theta \mapsto \rho$ and the parametric controller structure. The nonlinear function $q(t, x, \theta)$ and the adaptation gain, γ, a positive constant, are determined by the choice of algorithm.

Constructing the Parameter Adaptive Algorithm

We will restrict our discussion here to the gradient algorithm

$$\dot{\theta} = \gamma \phi \varepsilon \tag{2}$$

where $\phi(t) \in \mathbf{R}^p$ is referred to as the *regessor*, and $\varepsilon(t) \in \mathbf{R}$ as the *adaptation error*. In an *indirect* adaptive control, a typical error is the equation error

$$\varepsilon = y - \theta^T \phi \tag{3}$$

with control

$$u = -\rho^T \psi \tag{4}$$

and design rule

$$\rho = K(\theta) \tag{5}$$

where the elements of $\phi(t) \in \mathbf{R}^{n_\theta}$ consist of filtered versions of (y, u), and the elements of $\psi(t) \in \mathbf{R}^{n_\rho}$ consist of filtered versions of (y, u, r), where $r(t)$ is the reference command. In a *direct* adaptive control system we can take the error as

$$\varepsilon = y - y_m \tag{6}$$

with control

$$u = -\theta^T \phi \tag{7}$$

where $y_m(t)$ is the *reference model output.*

Goal of Adaptation

In general, adaptive algorithms share the same genesis and purpose no matter how they are constructed. Namely, to adjust $\theta(t)$ so that it asymptotically approaches a member of the set

$$\mathcal{F}_{opt} = \{\theta \in \mathbf{R}^{n_\theta} : \text{avg}\{\varepsilon^2\} \text{ is a minimum}\} \tag{8}$$

where $\text{avg}\{\cdot\}$ is defined by

$$\text{avg}\{X\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t)\, dt \tag{9}$$

In the ideal case, the parametrization can be selected so as to acheive what is called *perfect matching*, e.g., the set $\mathcal{F}_{opt}$ has a single member such that $\varepsilon(t) = 0$ or is white noise. Under more realistic conditions, the best that can happen

is that $\theta(t)$ asymptotically approaches a small neighborhood of

$$\mathcal{F}_* = \{\theta \in \mathbf{R}^{n_\theta} : \text{avg}\{\phi\varepsilon\} = 0, \text{avg}\{\phi\phi^T\} > 0\} \quad (10)$$

If for fixed θ, the regressor $\phi(t)$ is constructed to be identical with, or proportional to, $-\partial\varepsilon(t)/\partial\theta$, then $\mathcal{F}_*$ is the set of all *local minima* of $\text{avg}\{\varepsilon^2\}$. Since $\partial\varepsilon(t)/\partial\theta$ is a function of the true, but unknown plant, the regessor can at best be constructed as an approximation. Constructing the adaptive algorithm in this fashion has been referred to as the "MIT-Rule", see, e.g., Whitaker (1959) and Mareels et al.(1986).

From a practical point of view it is acceptable that the parameters approach and remain in a small neighborhood of $\mathcal{F}_*$, provided that members of this set also produce acceptable performance.

The Tuned System

Assuming this is so, let $\theta_* \in \mathbf{R}^{n_\theta}$ denote such a setting, of which there could be many. We refer to each θ_* as a *tuned parameter* and to the corresponding system

$$\dot{x}_* = A(\theta_*)x + B(\theta_*)w(t) \quad (11)$$

as the *tuned system* [see, e.g., Kosut and Friedlander (1985)]. Clearly, the tuned system is the same as system (1) but with $\theta(t)$ fixed at θ_*. We can now pose the following questions regarding the adaptive system (1):

(1) How do the tuned parameters depend on the exogenous inputs?

(2) Is the adaptive system stable in a neighborhood of the tuned system, i.e., are solutions (x, θ) stable near $(x_*(t), \theta_*)$? Furthermore, how small is this neighborhood?

(3) What is the region of attraction in (x, θ) to a small neighborhood of $(x_*(t), \theta_*)$?

(4) What is the rate of convergence to this small neighborhood of $(x_*(t), \theta_*)$?

Some of these questions can be answered by the method of averaging.

III. AVERAGING ANALYSIS

The method of averaging applies to a differential equation of the form

$$\dot{x} = \gamma f(t, x) \quad (12)$$

where γ is a positive constant. If γ is sufficiently small, then intuitively, the stability of the time-varying system (12) is inherited from the stability of the simpler *autonomous averaged system*

$$\dot{x} = \gamma \text{avg}\{f(\cdot, x)\} \quad (13)$$

More precisely, following Hale (1969) or Guckenheimer and Holmes (1983), we have:

Theorem 1 *Suppose that $f(t, x)$ is almost periodic in t and twice differentiable in x, for all x in a bounded set. Under these conditions:*

(i) If $x(t)$ and $x_a(t)$ are solutions of (12) and (13), respectively, then $\|x(0) - x_a(0)\| = \mathcal{O}(\gamma)$ implies that $\|x(t) - x_a(t)\| = \mathcal{O}(\gamma)$, $\forall t \in [0, \mathcal{O}(1/\gamma)]$.

(ii) Suppose there exists x° satisfying

$$\text{avg}\{f(\cdot, x^\circ)\} = 0 \quad (14)$$

Then, for all small $\gamma > 0$, there exists a unique solution $x_\gamma(t)$ of (12) with the following properties:

(ii-a) As $\gamma \to 0$, $x_\gamma(t) \to x^\circ$

(ii-b) $x_\gamma(t)$ is uniformly asymptotically stable (u.a.s.) if

$$\max_i \text{Re}\,\lambda_i[B(x^\circ)] < 0 \quad (15)$$

(ii-c) $x_\gamma(t)$ is unstable if

$$\max_i \text{Re}\,\lambda_i[B(x^\circ)] > 0 \quad (16)$$

where $x \mapsto B(x)$ is the matrix function

$$B(x) = \frac{\partial}{\partial x}\text{avg}\{f(\cdot, x)\} \quad (17)$$

In order to apply the above result to the adaptive system (1) it is first neccesary to make a transformation of variables, so that the resulting system has the appropriate form as expressed by (12). The transformation is referred to as a time-scale decomposition.

Time-Scale Decomposition

From the previous discussions about the origins of the adaptive algorithm, we are clearly interested in the behavior of the adaptive system in the neighborhood of the tuned system. But the tuned constant parameter setting θ_* is not known before hand. Hence, following the proceedures given in Anderson et al.(1986), we study the behavior of (1) in the neighborhood of *all* constant parameter solutions. For this purpose, let $\bar{x}(t, \theta)$ denote the state $x(t)$ when $\gamma = 0$. We refer to $\bar{x}(t, \theta)$ as the *frozen parameter system state*, or *frozen state* for short. Hence, for each $\theta \in \mathbf{R}^{n_\theta}$, $\bar{x}(t, \theta)$ satisfies the *partial* differential equation

$$\partial\bar{x}/\partial t = A(\theta)\bar{x} + B(\theta)w(t) \quad (18)$$

By introducing the error state

$$\eta(t) = x(t) - \bar{x}(t, \theta(t)) \quad (19)$$

the (x, θ)-system of (1) can be transformed into the equivalent (η, θ)-system:

$$\begin{aligned} \dot{\theta} &= \gamma f(t, \theta, \eta) \\ \dot{\eta} &= A(\theta)\eta - \gamma g(t, \theta, \eta) \end{aligned} \quad (20)$$

where the functions f and g are given by

$$\begin{aligned} f(t,\theta,\eta) &= g(\bar{x}(t,\theta)+\eta) \\ g(t,\theta,\eta) &= [\partial\bar{x}(t,\theta)/\partial\theta]f(t,\theta,\eta) \end{aligned} \tag{21}$$

The transformation of (1) into (20) is referred to as a *time-scale decomposition*, because for small γ, $\theta(t)$ changes much more slowly than $\eta(t)$. The averaged system which describes the parameter trajectories for small γ is then

$$\dot{\theta} = \gamma \text{avg}\{f(\cdot,\theta,0)\} \tag{22}$$

Using Theorem 1, the stability analysis of the adaptive system can be divided into answering questions about its asymptotic and transient characteristics, such as:

(1) *Asymptotic analysis*: What are the stability properties of (20) in the neighborhood of $(\theta,\eta) = (\theta_*,0)$?

(2) *Transient analysis*: What is the rate of convergence and region of attraction of (20) to a small neighborhood of $(\theta_*,0)$?

Asymptotic Analysis

The definition of the tuned system (x_*,θ_*) as well as the stability of the adaptive system in the neighborhood of the tuned system can be answered by Theorem 4.2 in Anderson et al.(1986) or Section 3 in Bodson et al.(1985). The flavor of these results, which follows from Theorem 1, can be stated as follows:

Theorem 2 *Suppose that the right hand sides in (1) are almost periodic in t and twice differentiable in both x and θ, for all x and θ in bounded sets. Under these conditions:*

(i) If $\theta_a(t)$ is a solution of (22), and if $\theta_a(0) = \theta(0), \bar{x}(0,\theta(0)) = x(0)$, then $\forall t \in [0,\mathcal{O}(1/\gamma)]$, $\|\theta(t) - \theta_a(t)\| = \mathcal{O}(\gamma)$ and $\|x(t) - x(t,\theta_a(t))\| = \mathcal{O}(\gamma)$.

(ii) Let the tuned parameter set be defined as those $\theta_* \in \mathbf{R}^{n_\theta}$ which satisfy

$$\begin{aligned} &\text{avg}\{f(\cdot,\theta_*,0)\} = 0 \\ &\text{Re}\,\lambda[A(\theta_*)] < 0 \end{aligned} \tag{23}$$

Then, for all small $\gamma > 0$, there exists a unique almost periodic solution $\theta_\gamma(t), x_\gamma(t)$ of (1) with the following properties:

(ii-a) As $\gamma \to 0$, $\theta_\gamma(t) \to \theta_*$ and $x_\gamma(t) \to x_*(t)$

(ii-b) $\theta_\gamma(t), x_\gamma(t)$ is uniformly asymptotically stable (u.a.s.) if

$$\max_i \text{Re}\,\lambda_i[R(\theta_*)] > 0 \tag{24}$$

(ii-c) $\theta_\gamma(t), x_\gamma(t)$ is unstable if

$$\max_i \text{Re}\,\lambda_i[R(\theta_*)] < 0 \tag{25}$$

where $\theta \mapsto R(\theta)$ is the matrix function

$$R(\theta) = -\frac{\partial}{\partial\theta}\text{avg}\{f(\cdot,\theta,0)\} \tag{26}$$

The sharp stability-instability boundary expressed by (24) and (25) allows not only for an assessment of a particular design, but also indicates how to modify and improve the algorithm. This is easier to illustate by using a specific application. Consider the direct adaptive control system (2),(6)-(7). The matrix $R(\theta_*)$ required in Theorem 2 is then

$$R(\theta_*) = \text{avg}\{\phi_*(H_{\epsilon v}\phi_*)^T + \epsilon_*(H_{\phi v}\phi_*)^T\} \tag{27}$$

where, referring to Figure 1, $H_{\epsilon v}$ and $H_{\phi v}$ are the *closed-loop* transfer functions with $\theta(t) = \theta_*$, from an exogenous input $v(t)$ inserted at the control input to the adaptation error ϵ and the regressor ϕ, respectively. Also, $\epsilon_*(t)$ is the tuned error, which in the ideal case would be white noise independent of $(H_{\phi v}\phi_*)(t)$, and hence, the second term above is zero. In general, $\epsilon_*(t)$ is small, and so the second term can be neglected.

Suppose that $\phi_*(t)$ is almost periodic with Fourier series representation

$$\phi_*(t) = \sum_{\omega\in\Omega} \alpha(\omega)e^{-j\omega t} \tag{28}$$

where Ω is the set of distinct Fourier frequencies and $\alpha(\omega)$ the corresponding coefficients. Then, with $\epsilon_*(t)$ small,

$$\begin{aligned} R(\theta_*) &\approx \sum_{\omega\in\Omega} \Phi(\omega)H_{\epsilon v}(-j\omega) \\ \Phi(\omega) &= \alpha(\omega)\bar{\alpha}^T(\omega) \end{aligned} \tag{29}$$

Observe that for (24) to hold it is necessary that $R(\theta_*)$ is non- singular, or eqivalently

$$\sum_{\omega\in\Omega} \Phi(\omega) > 0 \tag{30}$$

This is the usual condition for $\phi_*(t)$ to be *persistently exciting*, sometimes referred to as the condition for uniform identifiability of the parameters.

In the indirect case (2),(3)-(4), the expression for $R(\theta_*)$ is given by

$$R(\theta_*) = \text{avg}\{\phi_*\phi_*^T + [\phi_*(H_{\epsilon v}\psi_*)^T + \phi_*(H_{\psi v}\psi_*^T)]\frac{\partial\rho}{\partial\theta}(\theta_*)\} \tag{31}$$

where $H_{\epsilon v}, H_{\psi v}$ are closed loop transfer functions defined in the same sense as before. However, in the ideal indirect case, with no unmodeled dynamics, both $H_{\epsilon v}$ and $H_{\psi v}$ vanish. A derivation of the above expression and further discussion can be found in Philips et al.(1987).

We also remark that for any adaptive system, the local stability test matrix $R(\theta)$ can be evaluated at any candidiate value of θ, not just θ_* as defined in Theorem 2. In this case we can insure stability in a small neighborhhod of the

candidate parameter value, provided that the adaptation error $\varepsilon(t)$ is sufficiently small at the candidate parameter value, see, e.g., Anderson et al.(1986).

Signal Dependent Positivity Condition

Since for any square matrix M, we have the inequality $\mathrm{Re}\,\lambda(M) \leq (1/2)\lambda(M + M^T)$, it follows that a sufficient condition for (24) to hold, with $R(\theta_*)$ given by (29), is that

$$\min_i \lambda_i \{ \sum_{\omega \in \Omega} \Phi(\omega) \mathrm{Re}\, H_{\varepsilon v}(j\omega) \} > 0 \qquad (32)$$

This is the local stability condition first stated by Riedle and Kokotovic (1985), and referred to as a *signal dependent positivity condition*. To provide some interpretation, suppose that $H_{\varepsilon v}(s)$ is stricly positive real (SPR), [1]

$$\mathrm{Re}\, H_{\varepsilon v}(j\omega) \geq \delta |H_{\varepsilon v}(j\omega)|^2 \,, \forall \omega \in \mathbf{R} \qquad (33)$$

where δ is a positive constant. Condition (32) is clearly significantly less restrictive than the SPR condition (33), which restricts $\mathrm{Re}\, H_{\varepsilon v}(j\omega)$ at each frequency, whereas (32) is a weighted sum of each $\mathrm{Re}\, H_{\varepsilon v}(j\omega)$ contribution, hence the terminology mentioned above. Local stability is assured as long as the contributions from those frequencies where $\mathrm{Re}\, H_{\varepsilon v}(j\omega) < 0$ are dominated by those where $\mathrm{Re}\, H_{\varepsilon v}(j\omega) > 0$. Although (32) has a nice interpretation, in light of (24) it is not necessary to be so conservative. In fact, it is easy to construct an example where (32) fails but (24) holds nonetheless.

Transient Analysis

An understanding of the transient properties of the adaptive system requires answering the questions posed before, namely, determining the region of attraction to a small neighborhood of the tuned system, and the rate of convergence. The following result, based on averaging, can be found in Anderson et al.(1986).

Theorem 3 *Under the conditions stated in Theorem 2, if the initial parameter value θ_o is strictly inside a convex subset of the constant parameter stability set, then $\theta(t)$ will remain in that set and converge exponentially at a rate no slower than $\mathcal{O}(\gamma)$ to the u.a.s. almost periodic solution $\theta_\gamma(t)$ defined in Theorem 2.*

Some of the restrictions can be relaxed, see, e.g., Riedle and Kokotovic (1986) or Bodson et al.(1985). Specifically, the region of attraction can be relaxed to a compact subset of the constant parameter stability set; also, estimates of the rate of convergence far from the tuned setting do not have to be $\mathcal{O}(\gamma)$.

These type of results are pleasing from an intuitive point of view, thus providing a *qualitative* analysis. However, they do not apply when the parameters leave the constant parameter stability set, nor do they provide *quantitative* performance measures. In addition, we still require slow adaptation, which in itself is not too objectionable, perhaps only cautious, but again specific bounds on the adaption gain extracted from the theory tend to be extremely conservative. In the next section we show how it may be possible to remedy some of these difficulties

[1] The SPR condition is not just pulled out of the air, it arises as the condition to insure boundedness of the parameter estimates, provided there are no unmodeled dynamics or disturbances, see, e.g., Narendra et al.(1980). Essentially, the SPR condition forces the algorithm to never move "uphill" when searching for the minimum.

IV. FIXED POINT ANALYSIS

In this section we briefly discuss an alternative means for analyzing the adaptive system, which does not have to rely on averaging. We utilize the fixed-point theorem of Banach and Cacciopoli, referred to as the *contraction mapping principal*. [Actually, the adaptive system (1) can be analyzed by calling upon one of several fixed point theorems, Kosut and Bitmead (1986).]

Following Hale (1969), let $\mathcal{M}$ be a subset of a Banach space $\mathcal{B}$ with norm $\|\cdot\|$. If Γ is an operator mapping $\mathcal{M} \to \mathcal{B}$, then Γ is a *contraction on* $\mathcal{M}$ if there is a constant $\sigma \in [0,1)$ such that

$$\|\Gamma x - \Gamma y\| \leq \sigma \|x - y\| \,, \forall x, y \in \mathcal{M} \qquad (34)$$

The constant σ is referred to as the *contraction constant for Γ on $\mathcal{M}$*. A *fixed point* of $\Gamma : \mathcal{M} \mapsto \mathcal{M}$ is a point (function) $x \in \mathcal{M}$ such that $x = \Gamma x$. We can now state the

Contraction Mapping Principal *If $\mathcal{M}$ is a closed subset of a Banach space $\mathcal{B}$ and $\Gamma : \mathcal{M} \to \mathcal{M}$ is a contraction on $\mathcal{M}$, then Γ has a unique fixed point in $\mathcal{M}$.*

Referring to (20), we can take Γ as the mapping of functions $\bar{\theta}(t)$ into functions $\hat{\theta}(t)$ defined implicitly as follows:

$$\dot{\hat{\theta}} = \gamma f(t, \hat{\theta}, \bar{\eta}),\ \hat{\theta}(0) = \theta(0) \qquad (35)$$

$$\dot{\bar{\eta}} = A(\bar{\theta})\bar{\eta} - \gamma g(t, \bar{\theta}, \bar{\eta}),\ \eta(0) = 0 \qquad (36)$$

Observe that fixed points of Γ in $\mathcal{M}$, i.e., those functions $\theta \in \mathcal{M}$ which satisfy the operator equation

$$\theta = \Gamma\theta \qquad (37)$$

are solutions in $\mathcal{M}$ of the parameter trajectories of the adaptive system (20), or equivalently (1). For example, the results in Theorem 3 are arrived at by choosing

$$\mathcal{M} = \{\theta \in C[0,\infty) : \|\theta(t) - \theta_*\| \leq r_o + r_1 \exp(-\lambda t)\} \qquad (38)$$

where $C[0,\infty)$ is the Banach space of continuous bounded functions, and r_o, r_1, and λ are positive constants.

In the process of establishing that Γ is contractive on $\mathcal{M}$, we can utilize the method of averaging to establish the stability, near θ_*, of

$$\dot{\theta} = \gamma f(t, \theta, 0) \qquad (39)$$

This is the origin of condition (24). It is important to point out that neither averaging nor small γ is required to establish the stability of (39) near θ_*. For example, if the function $f(t, \theta, 0)$ is periodic in t uniformly for θ in a compact set, then stability of (39) near θ_* can be established by linearization and Floquet Theory, Hale (1969).

Condition (24) is then replaced by a weaker condition and the limitation on the allowable size of the adaptation gain is considerably reduced over that imposed by averaging theory. An early application of Floquet theory to the linearized adaptive system is reported in a paper by James (1971). A similar example is given in Mareels et al.(1986).

V. CONCLUDING REMARKS

We have briefly described some recent results, mostly contained in Anderson et al.(1986), which pertain to the stability analysis of adaptive systems. As indicated, these results, which are based on the classical method of averaging, can be extended to other than slow adaptation by calling upon one of several fixed point theorems. Since the use of such results is well beyond hand calculation it is imperitive to develop the means to numerically establish the contraction conditions. Observe that because the contraction analysis considers operators in Banach spaces, the same results apply to discrete-time or hybrid adaptive systems, i.e., any linear adaptive control of a linear plant. Extentions to adaptive nonlinear systems, although similar in principal to the present analysis, often turn out to have their own dominating idiosyncrasies, e.g., Workman et al.(1987).

REFERENCES

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, (1986), *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

B.D.O. Anderson (1985), "Adaptive Systems, Lack of Persistency of Excitation and Bursting Phenomena," *Automatica*, vol 21, no 3, pp 247- 258.

K.J.Astrom (1983),"Analysis of Rohrs Counter-Examples to Adaptive Control", *Proc. 22nd IEEE Conf. on Dec. and Contr.*, San Antonio, TX.

K.J.Astrom (1984),"Interactions Between Excitation and Unmodeled Dynamics in Adaptive Control",*Proc. 23rd IEEE Conf. on Dec. and Contr.*,Las Vegas, NV, pp.1276-1281.

M. Bodson, S. Sastry, B.D.O. Anderson, I. Mareels, and R.R. Bitmead, (1986), "Nonlinear Averaging Theorems, and the Determination of Parameter Convergence Rates in Adaptive Control", *Systems and Contr. Letters*, to appear.

J. Guckenheimer and P. Holmes, (1983), *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag.

J.K. Hale (1980), *Ordinary Differential Equations*, Kreiger, Molaban, FL, originally published (1969), Wiley (Interscience), New York.

D.J.G. James, (1971), "Stability of a Model Reference Control System", *AIAA Journal*, vol. 9, pp.950-952.

R.L. Kosut, B.D.O. Anderson, and I.M.Y. Mareels, (1987), "Stability Theory for Adaptive Systems: Methods of Averaging and Persistency of Excitation", *IEEE Trans. on Aut. Contr.*, Vol. AC-32, No. 1, pp. 20–34, Jan. 1987.

R.L. Kosut, I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, and C.R. Johnson, Jr., (1987),"Transient Analysis of Adaptive Control", *Proc. IFAC, 10th World Congress*, Munich, Germany, July 1987.

R.L. Kosut and R.R. Bitmead (1986), "Fixed Point Theorems for the Stability Analysis of Adaptive Systems", *Proc. IFAC Workshop on Adaptive Control*, Lund, Sweden.

R.L. Kosut and B. Friedlander (1985), "Robust Adaptive Control: Conditions for Global Stability", *IEEE Trans. on Aut. Contr.*, AC-30(7):610–624.

L. Ljung and T. Soderstrom, (1983), *Theory and Practice of Recursive Identification*, MIT Press, Cambridge, Mass.

I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, M. Bodson, and S.S. Sastry, (1986), "Revisiting the MIT-Rule for Adaptive Control", *Proc. IFAC Workshop on Adaptive Control*, Lund, Sweden.

K.S. Narendra, Y.H. Lin, and L.S. Valavani, (1980), "Stable Adaptive Control Design, Part II: Proof of Stability", *IEEE Trans. Aut. Control*, vol AC -25, no. 3, pp.440-449.

S. Philips, R.L. Kosut, and G.F. Franklin, (1987), "An Averaging Analysis of Discrete-Time Indirect Adaptive Control", *ISL Report*, Stanford University, August, 1987.

B.D. Riedle and P.V. Kototovic (1986), "Integral Manifolds of Slow Adaptation", *IEEE Trans. Aut. Control*, Vol. 31, No. 4, pp. 316–324, April 1986.

B.D. Riedle and P.V. Kokotovic (1985), "A Stability-Instability Boundary for Disturbance-Free Slow Adaptation and Unmodeled Dynamics", *IEEE Trans. on Aut. Contr.*, AC-30:1027–1030.

C.E. Rohrs, L.S. Valavani, M. Athans, and G. Stein (1985),"Robustness of Continuous Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics",*IEEE Trans. Aut. Contr.*, AC-30(9):881-889, Sept.

M.G.Safonov, A.J.Laub, and G.L.Hartman (1981),"Feedback Properties of Multivariable Systems: The Role and Use of the Return Difference Matrix", *IEEE Trans. Aut. Contr.*, Vol. 26(1):47-65, Feb. 1981.

M. Vidyasagar (1985), *Control system Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

H.P. Whitaker (1959), "An adaptive System for Control of the Dynamics Performance of Aircraft and Spacecraft", *Inst. Aeronautical Sciences*, Paper 59-100.

M.Workman, R.L.Kosut, and G.F.Franklin (1987),"Adaptive Time-Optimal Control: Continuous Time Case", *Proc. 1987 ACC*, Minneapolis, MN, June,1987.

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Phila., PA

CONDITIONS FOR CONVERGENCE AND DIVERGENCE OF PARAMETER ADAPTIVE LINEAR SYSTEMS

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Abstract - **A generic parameter adaptive system is proposed and analyzed for the case where the system is linear time invariant whenever the parameters are held fixed. Conditions for local convergence and divergence are obtained.**

I. INTRODUCTION

Uncertainty underlies the reason for adaptation. Some physical processes defy practical mathematical modeling, whereas others, with a well defined structure, have uncertain parameters. In both of these instances it is often the case that filtering, prediction, and control systems can be designed with adjustable parameters which can be tuned so as to accomadate the system uncertainty.

At the present time we stand at the beginning stages in the development of theory which is directly applicable to adaptive systems. To cite one example, in the past few years there has been vigorous activity and debate on the question of the robustness of adaptive control, i.e., what happens when ideal conditions are violated, as would be expected in practice. For example, simulations of simple systems under apparently minor non-ideal conditions have shown degraded performance and even instabilities, e.g., Rohrs et al.(1985). Although there is still not as yet a complete theory for the stability of adaptive systems, many of the issues can now be addressed.

In this paper we examine the stability of adaptive systems from the point of view of parameter convergence or divergence. We review and extend some of the results which are contained in the recently published textbook by Anderson et al.(1986). These results arise from the application of some of the classical methods for analyzing differential equations, e.g., linearization, the method of averaging, and Lyapunov second method, see, e.g., Hale(1969). The material in the text represents modifications and refinements of earlier work, specifically: Astrom (1983, 1984) showing how the method of averaging explains instabilities and drift; Reidle and Kokotovic (1985, 1986) on slow adaptation and the integral manifold; Kosut, Anderson, and Mareels(1987) on the relation between averaging and persistent excitation; Bodson et al.(1986) on nonlinear averaging analysis and determining the rate of convergence; and Kosut and Anderson (1986), Kosut and Johnson(1984) on linearization and local stability. Stochastic averaging methods have also been employed to study the asymptotic parameter trajectories of recursive parameter estimation algorithms in system identification methods, e.g., Ljung and Soderstrom(1983).

Averaging: Uses and Limitations

Although the method of averaging provides a great deal of insight into the behavior of adaptive systems, it is not a panacea. In the first place, the method of averaging requires slow adaptation which can be counter-productive in some instances because performance can be below par for the long period of time it takes for the parameters to adjust. Secondly, the results are valid only when the parameters are restricted to a subset of the constant parameter stability set. Projection techniques can be employed to restrict the parameters as required, but in many practical cases this is not feasible nor even necessary, e.g., in output error system identification and adaptive control, momentary unstable parameter settings can induce very rapid learning. However, the exact mechanism is not well understood.

Another area of concern when using the method of averaging as an analysis tool, is that the speed of adaptation required to satisfy the *theoretical* conditions is most often far below that as determined from simulations. Thus, although slow adaptation allows for an analysis which provides quantitative measures of stability-instability boundaries, rate of convergence, and robustness to unmodeled effects, there are

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inherent restrictions in the analysis.

Transient Analysis

To remove these restrictions requires understanding the *transient behavior* of adaptive systems. Some preliminary results are reported in Kosut et al. (1987) and Kosut and Bitmead (1986). The tools for analysis involve a combination of small gain theory, passivity, and the method of averaging, with these all linked together by the Contraction Mapping Principal. Some of these ideas will be described and expanded upon here.

Beyond Hand Calculations

Although each of these tools, in principal, involves straightforward calculations, even simple examples can just barely be worked out by hand. It is clear that the level of complexity of a realistic adaptive system is well beyond hand calculation. Hence, in order for any of the above mentioned analytic methods to be of practical benifit, it is imperitve to develop "user-friendly" software tools which provide the requisite nonlinear analysis.

At the present time, aside from simulation capbility, there are no available software tools for dealing principally with adaptive systems, and certainly none for more general nonlinear systems. This is a research issue in both mathematics and computation, and it is one that is essential to a continuing study of adaptive systems. This point will not be pursued here, but it is sometimes an overlooked research issue, and thus warrents more than a passing remark.

II. LINEAR ADAPTIVE SYSTEMS

Adaptive System Structure

The adaptive system we will examine is shown in Figure 2-1. This system structure is valid for most parameter adaptive filtering, prediction, and control systems, be they continuous, discrete, or hybrid [see, e.g., Kosut and Bitmead(1986), Ljung and Soderstrom(1983)].

In this system $u(t)$ and $y(t)$ are the vectors of measured inputs and outputs, $w(t)$ is a vector of exogenous inputs, i.e., references, disturbances, and noise sources. The vector of adaptive parameters is $\theta(t)$ with initial value θ_o. The adaptive parameter vector is mapped into a "control" parameter vector $\rho(t)$ by some design rule, typically an implicitly defined memoryless nonlinear function . For example, in control system synthesis, one can use any number of model (parameter) based methods, see e.g., Safonov et al.(1981) on LQG based designs or Vidyasagar(1985) on the stable factorization approach. In system identification, on

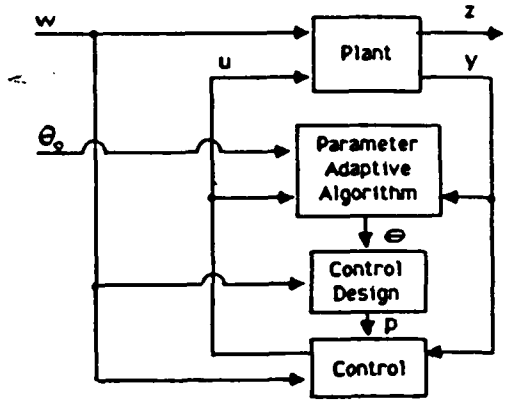

Figure 2-1. Parameter adaptive control

the other hand, we normally have a the simple design rule $\theta = \rho$. A more extensive display of how the various standard adaptive identification and control systems fit into the structure of Figure 2-1 is provided in Anderson et al.(1986).

Parameter Adaptive Algorithm

In order to illustrate the basic ideas, we will assume here that the parameter adaptive algorithm is contiuously adjusted. In practice, parameter adjustments would be either at discrete times or as in the case of adaptive control, the control signal is the output of a digital computer.

We will also make the simplifying assumption that the plant is linear-time-invariant with a rational proper transfer function, and for fixed values of the control parameter vector ρ, so is the block labled control. Then, the parameter adaptive system of Figure 2-1 may be described by the coupled set of ordinary differential equations

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)w(t) \\ \dot{\theta} &= \gamma(t)q(t,x,\theta) \end{aligned} \tag{2.1}$$

where $\theta(t) \in \mathbf{R}^p$ is the adaptive parameter, and $x(t) \in \mathbf{R}^n$ is the system state, consisting of plant, controller, and filter states. The matrix functions $A(\theta), B(\theta)$ are determined by the design rule $\theta \mapsto \rho$ and the parametric controller structure. The nonlinear function $q(t,x,\theta)$ is determined by the choice of algorithm, and $\gamma(t)$ is the speed of adjustment, often referred to as the adaptation gain. For example, with a simple gradient algorithm $\gamma(t) = \gamma$, a positive constant, whereas with a recursive least squares (RLS) type of algorithm $\gamma(t) = 1/t$. Most algorithms are never used in these

simplified forms. Modifications include gain normalization, dead-zones, leakage, projection, and covariance resetting, to name a few. The reasons for the modifications arise from a priori knowlege of parameter ranges and the desire to add some robustness to the algorithm. Obviously there are a myriad of possibilities.

In general, adaptive algorithms share a similar genisis and purpose, namely to adjust $\theta(t)$ so that it asymptotically approaches a member of the set

$$\mathcal{F}_{opt} = \{\theta \in \mathbf{R}^p : \text{avg}[\varepsilon^2] \text{ is a minimum}\} \tag{2.2}$$

where $\text{avg}(\cdot)$ is defined by

$$\text{avg}(X) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)\, dt \tag{2.3}$$

In the ideal case, the parametrization can be selected so as to acheive what is called *perfect matching*, i.e., for some restricted class of exogenous inputs (usually no disturbances), and a sufficient number of parameters such that there are no unmodeled dynamics, the set $\mathcal{F}_{opt}$ has a single member such that $\varepsilon(t) = 0$. Under more realistic conditions, the best that can happen is that $\theta(t)$ asymptotically approaches a small neighborhood of

$$\mathcal{F}_* = \{\theta \in \mathbf{R}^p : \text{avg}[\phi\varepsilon] = 0\} \tag{2.4}$$

If for fixed θ, the regressor $\phi(t)$ is constructed to be identical with $-\partial\varepsilon(t)/\partial\theta$, then $\mathcal{F}_*$ is the set of all *local minima* of $\text{avg}[\varepsilon^2]$. Since $\partial\varepsilon(t)/\partial\theta$ is a function of the true, but unknown plant, the regessor can at best be constructed as an approximation. From a practical point of view it is acceptable that the parameters approach and remain in a small neighborhood of $\mathcal{F}_*$, provided that members of this set also produce acceptable performance.

The Tuned System

Assuming this is so, let $\theta_* \in \mathbf{R}^p$ denote such a setting, of which there could be many. We refer to each θ_* as a *tuned parameter* and to the corresponding system

$$\begin{aligned} \dot{x}_* &= A(\theta_*)x + B(\theta_*)w(t) \\ \varepsilon_* &= c^T x_* \\ \phi_* &= Dx_* \end{aligned} \tag{2.5}$$

as the *tuned system* [see, e.g., Kosut and Friedlander(1985)]. Clearly (2.5) is the same as system (2.1) but with $\theta(t)$ fixed at θ_*. We can now pose the following questions regarding the adaptive system (2.1):

(1) How do the tuned parameters depend on the exogenous inputs?

(2) Is the adaptive system stable in a neighborhood of the tuned system, i.e., are solutions (x, θ) stable near (x_*, θ_*)? Furthermore, how small is this neighborhood?

(3) What is the region of attraction in (x, θ) to a small neighborhood of (x_*, θ_*)?

(4) What is the rate of convergence to this small neighborhood of (x_*, θ_*)?

Some of these questions can be answered by the method of averaging.

III. AVERAGING ANALYSIS

The classical method of averaging applies to a differential equation of the form

$$\dot{x} = \gamma f(t, x)\ ,\ \gamma > 0 \tag{3.1}$$

where γ is a positive constant. If γ is sufficiently small then under suitable regularity conditions on $f(t, x)$, the stability of the time-varying system (3.1) is inherited from the stability of the simpler autonomous system

$$\dot{x} = \gamma f_a(x) \tag{3.2}$$

where

$$\begin{aligned} f_a(x) &= \text{avg} f(\cdot, x) \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)\, dt \end{aligned} \tag{3.3}$$

This system is referred to as the *averaged* system. Observe that for $f_a(x)$ to exist, some other restrictions must also apply to $f(t, x)$, e.g., $f(t, x)$ is almost periodic in t uniformly for x in a compact set; although weaker conditions can be stated.

The above type of result is certainly expected on intuitive grounds, provided that $f(t, x)$ is smooth enough and the average value $f_a(x)$ exists. A precise formulation of the stability-instability conditions is given in Theorem V.3.1, Hale(1969).

In order to apply the above result to the adaptive system (2.1) it is first neccesary to make a transformation of variables, so that the resulting system has the appropriate form

as expressed by (3.1). The transformation is referred to as a time-scale decomposition.

Time-Scale Decomposition

From the previous discussions about the origins of the adaptive algorithm, we are clearly interested in the behavior of the adaptive system in the neighborhood of the tuned system. But the tuned constant parameter setting θ_* is not known before hand. Hence, following the proceedures given in Anderson et al.(1986), we study the behavior of (2.1) in the neighborhood of *all* constant parameter solutions. For this purpose, let $\nu(t,\theta)$ denote the state $x(t)$ when $\gamma(t) = 0$. We refer to $\nu(t,\theta)$ as the *frozen parameter system state*, or *frozen state* for short. Hence, for each $\theta \in \mathbf{R}^p$, $\nu(t,\theta)$ satisfies the *partial* differential equation

$$\partial\nu/\partial t = A(\theta)\nu + B(\theta)w(t) \tag{3.4}$$

By introducing the error state

$$\eta(t) = x(t) - \nu(t,\theta(t)) \tag{3.5}$$

the (x,θ)-system of (2.1) can be transformed into the equivalent (η,θ)-system:

$$\begin{aligned}\dot\theta &= \gamma(t)f(t,\theta,\eta)\\ \dot\eta &= A(\theta)\eta - \gamma(t)g(t,\theta,\eta)\end{aligned} \tag{3.6}$$

where the functions f and g are given by

$$\begin{aligned}f(t,\theta,\eta) &= q(\nu(t,\theta)+\eta)\\ g(t,\theta,\eta) &= [\partial\nu(t,\theta)/\partial\theta]f(t,\theta,\eta)\end{aligned} \tag{3.7}$$

The transformation of (2.1) into (3.6) is referred to as a *time-scale decomposition*, because in general, $\theta(t)$ changes much more slowly than $\eta(t)$. This is certainly the case whenever the adaptation gain, $\gamma(t)$, is small. Observe that when $\gamma(t) = 1/t$, there is always some time $t_o > 0$ for which $\gamma(t)$ is as small as necessary. But, whenever $\theta(t)$ is near convergence, it is also changing slowly, and hence, we would expect the averaging results to apply even though the adaptation gain may not be small. We will return to this point later.

For now, let us assume that the adaptation gain is constant, but small, i.e., $\gamma(t) = \gamma$ where γ is a small positive number. Since the (η,θ)-system of (3.6) is equivalent to the original (x,θ)-system of (2.1), the answers to the questions posed before will involve the analysis of (3.6). In particular, the analysis can divided into answering questions about its asymptotic and transient characteristics.

Asymptotic Analysis

The definition of the tuned system (x_*,θ_*) as well as the stability of the adaptive system in the neighborhood of the tuned system can be answered by Theorem 4.2 in Anderson et al.(1986) or Section 3 in Bodson et al.(1985). The flavor of these results can be stated as follows:

Theorem 1 *Let the tuned parameter set be defined as those $\theta_* \in \mathbf{R}^p$ which satisfy*

$$\begin{aligned}&\mathrm{avg}[f(\cdot,\theta_*,0)] = 0\\ &\mathrm{Re}\,\lambda[A(\theta_*)] < 0\end{aligned} \tag{3.8}$$

If the functions $f(t,\theta,\eta)$ and $g(t,\theta,\eta)$ are sufficiently smooth, almost periodic in t uniformly for θ,η in compact sets, then for a sufficiently small adaptation gain γ, and a sufficiently small peak value of the tuned error signal $\varepsilon_*(t)$, solutions of (2.1) originating in a small neighborhood of the tuned system (x_*,θ_*) will remain there if

$$\max_i \mathrm{Re}\,\lambda_i[B(\theta_*)] < 0 \tag{3.9}$$

and, moreover, will leave there if

$$\max_i \mathrm{Re}\,\lambda_i[B(\theta_*)] > 0 \tag{3.10}$$

where the matrix function $\theta \mapsto B(\theta)$ is given by

$$B(\theta) = \frac{\partial}{\partial\theta}\mathrm{avg}[f(\cdot,\theta,0)] \tag{3.11}$$

In addition, under the conditions stated above, if (3.9) holds then

$$\begin{aligned}\limsup_{t\to\infty}\|\theta(t)-\theta_*\| &= O(\limsup_{t\to\infty}|\varepsilon_*(t)|) + O(\gamma)\\ \limsup_{t\to\infty}\|\eta(t)\| &= O(\gamma)\end{aligned} \tag{3.12}$$

The type of smoothness conditions required of the functions $f(t,\theta,\eta)$ and $g(t,\theta,\eta)$ is not severe, e.g., continuity in t and Lipschitz continuity in θ,η for θ,η in compact sets.

The sharp stability-instability boundary expressed by (3.9) and (3.10) allows not only for an assessment of a particular design, but also indicates how to modify and improve the algorithm.

It is possible to obtain specific expressions for the right hand sides of (3.12), but these are usually too coarse to be of practical quantitative value. In the next section we

will discuss the underlying method for obtaining the above results and show that, in the first place, averaging is not necessary, and secondly the conservatism in the bounds can be reduced significantly. First, however, we address the transient characteristics.

Transient Analysis

An understanding of the transient properties of the adaptive system requires answering the questions posed before, namely, determining the region of attraction to a small neighborhood of the tuned system, and the rate of convergence. The folowing result, based on averaging, can be found in Anderson et al.(1986).

Theorem 2 *Under the conditions stated in Theorem 1, if the initial parameter value θ_o is strictly inside a convex subset of the constant parameter stability set, then $\theta(t)$ will remain in that set and converge exponentially at a rate no slower than $O(\gamma)$ to the small neighborhood of the tuned system defined in Theorem 1.*

Some of the restrictions can be relaxed, see, e.g., Riedle and Kokotovic(1986) or Bodson et al.(1985). Specifically, the region of attraction can be relaxed to a compact subset of the constant parameter stability set; also, estimates of the rate of convergence far from the tuned setting do not have to be $O(\gamma)$.

These type of results are pleasing from an intuitive point of view, thus providing a *qualitative* analysis. However, they do not apply when the parameters leave the constant parameter stability set, nor do they provide *quantitative* performance measures. In addition, we still require slow adaptation, which in itself is not too objectionable, perhaps only cautious, but again specific bounds on the adaption gain extracted from the theory tend to be extremely conservative. In the next section we show how it may be possible to remedy some of these difficulties

IV. FIXED POINT ANALYSIS

Here we will descibe a general nonlinear analysis tool, namely the fixed-point theorem of Banach and Cacciopoli, referred to as the *contraction mapping principal*. Among other possibilities, it enables one to compute the rate of convergence and region of attraction for the adaptive system. [Actually, the adaptive system as represented by (3.6) can be analyzed by calling upon one of several fixed point theorems, Kosut and Bitmead(1986).]

Following Hale(1969), let $\mathcal{M}$ be a subset of a Banach space $\mathcal{B}$ with norm $\|\cdot\|$. If Γ is an operator mapping $\mathcal{M} \to \mathcal{B}$, then Γ is a *contraction on* $\mathcal{M}$ if there is a constant $\rho \in [0,1)$ such that

$$\|\Gamma x - \Gamma y\| \leq \rho \|x - y\| \ , \forall x, y \in \mathcal{M} \tag{4.1}$$

The constant ρ is referred to as the *contraction constant for Γ on $\mathcal{M}$*. A *fixed point* of $\Gamma : \mathcal{M} \mapsto \mathcal{M}$ is a point (function) $x \in \mathcal{M}$ such that $x = \Gamma x$. We can now state the

Contraction Mapping Principal *If $\mathcal{M}$ is a closed subset of a Banach space $\mathcal{B}$ and $\Gamma : \mathcal{M} \to \mathcal{M}$ is a contraction on $\mathcal{M}$, then Γ has a unique fixed point in $\mathcal{M}$.*

Referring to (3.6), we can take Γ as the mapping of functions $\bar{\theta}(t)$ into functions $\theta(t)$ defined implicitly as follows:

$$\begin{aligned} \dot{\theta} &= \gamma f(t, \theta, \bar{\eta}) \\ \dot{\bar{\eta}} &= A(\bar{\theta})\bar{\eta} - \gamma g(t, \bar{\theta}, \bar{\eta}) \end{aligned} \tag{4.2}$$

Observe that fixed points of Γ in $\mathcal{M}$, i.e., those functions $\theta \in \mathcal{M}$ which satisfy the operator equation

$$\theta = \Gamma\theta \tag{4.3}$$

are solutions in $\mathcal{M}$ of the parameter trajectories of the adaptive system (3.6), or equivalently (2.1). For example, the results (3.9)-(3.12) are arrived at by choosing

$$\mathcal{M} = \{\theta \in C[0,\infty) : \|\theta(t) - \theta_*\| \leq r_o + r_1 \exp(-\sigma t)\} \tag{4.4}$$

where $C[0,\infty)$ is the Banach space of continuous bounded functions, and r_o, r_1, and σ are positive constants.

In the process of establishing that Γ is contractive on $\mathcal{M}$, we utilize the method of averaging to establish the stability, near θ_*, of

$$\dot{\theta} = \gamma f(t, \theta, 0) \tag{4.5}$$

This is the origin of condition (3.9). It is important to point out that neither averaging nor small γ is required to establish the stability of (4.5) near θ_*. For example, if the function $f(t, \theta, 0)$ is periodic in t uniformly for θ in a compact set, then stability of (4.5) near θ_* can be established by linearization and Floquet Theory, Hale(1969). Condition (3.9) is then replaced by a weaker condition and the limitation on the allowable size of the adaptation gain is

considerably reduced over that imposed by averaging theory. An early application of Floquet theory is reported in a paper by James (1971), although it was not carried over to the nonlinear adaptive system.

V. CONCLUDING REMARKS

We have briefly described some recent results, mostly contained in Anderson et al.(1986), which pertain to the stability analysis of adaptive systems. As indicated, these results, which are based on the classical method of averaging, can be extended to other than slow adaptation by calling upon one of several fixed point theorems. Since the use of such results is well beyond hand calculation it is imperitive to develop the means to numerically establish the contraction conditions. Establishing such conditions can be viewed as a *canonical problem* in analyzing the stability properties of adaptive systems. Observe that because the contraction analysis considers operators in Banach spaces, the same results apply to discrete-time or hybrid adaptive systems, i.e., any linear adaptive control of a linear plant. Extentions to adaptive nonlinear systems, although similar in principal to the present analysis, often turn out to have their own dominating idiosyncrasies, e.g., Workman et al.(1987).

REFERENCES

B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, (1986),*Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

K.J.Astrom (1983),"Analysis of Rohrs Counter-Examples to Adaptive Control", *Proc. 22nd IEEE Conf. on Dec. and Contr.*, San Antonio, TX.

K.J.Astrom (1984),"Interactions Between Excitation and Unmodeled Dynamics in Adaptive Control",*Proc. 23rd IEEE Conf. on Dec. and Contr.*,Las Vegas, NV, pp.1276-1281.

M. Bodson, S. Sastry, B.D.O. Anderson, I. Mareels, and R.R. Bitmead, (1986), "Nonlinear Averaging Theorems, and the Determination of Parameter Convergence Rates in Adaptive Control", *Systems and Contr. Letters*, to appear.

J.K. Hale (1980), *Ordinary Differential Equations*, Kreiger, Molaban, FL, originally published (1969), Wiley (Interscience), New York.

R.L. Kosut, B.D.O. Anderson, and I.M.Y. Mareels,(1987), "Stability Theory for Adaptive Systems: Methods of Averaging and Persistency of Excitation", *IEEE Trans. on Aut. Contr.*, Vol. AC-32, No. 1, pp. 20-34, Jan. 1987.

R.L. Kosut, "Floquet Anaysis of Adaptive Systems"(1987), *Fifth Yale Workshop on Applications of Adaptive Systems Theory*, New Haven, CT, May 1987.

R.L.Kosut, I.M.Y.Mareels, B.D.O.Anderson, R.R.Bitmead, andC.R. Johnson, Jr., (1987),"Transient Analysis of Adaptive Control", to be presented at the *IFAC, 10th World Congress*, Munich, Germany, July 1987.

R.L. Kosut and R.R. Bitmead (1986), "Fixed Point Theorems for the Stability Analysis of Adaptive Systems", *IFAC Workshop on Adaptive Control*, AC-30:834.

R.L. Kosut and B.D.O. Anderson,(1986), "Local Stability Analysis for a Class of Adaptive Systems", *IEEE Trans. Auto. Contr.*, Jan. 1986.

R.L. Kosut and B. Friedlander (1985), "Robust Adaptive Control: Conditions for Global Stability", *IEEE Trans. on Aut. Contr.*, AC-30(7):610-624.

R.L. Kosut and C.R. Johnson, Jr. (1984), "An Input-Output View of Robustness in Adaptive Control", *Automatica*, 20(5)569-581.

L. Ljung and T. Soderstrom (1983), *Theory and Practice of Recursive Identification*, MIT Press.

B.D. Riedle and P.V. Kototovic (1986), "Integral Manifolds of Slow Adaptation", *IEEE Trans. Aut. Control*, Vol. 31, No. 4, pp. 316-324, April 1986.

B.D. Riedle and P.V. Kokotovic (1985), "A Stability-Instability Boundary for Disturbance-Free Slow Adaptation and Unmodeled Dynamics", *IEEE Trans. on Aut. Contr.*, AC-30:1027-1030.

C.E.Rohrs, L.S.Valavani, M.Athans, and G.Stein (1985),"Robustness of Continuous Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics",*IEEE Trans. Aut. Contr.*, AC-30(9):881-889, Sept.

M.G.Safonov, A.J.Laub, and G.L.Hartman(1981),"Feedback Properties of Multivariable Systems: The Role and Use of the Return Difference Matrix", *IEEE Trans. Aut. Contr.*, Vol. 26(1):47-65, Feb. 1981.

M. Vidyasagar (1985), *Control system Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

M.Workman, R.L.Kosut, and G.F.Franklin(1987),"Adaptive Time-Optimal Control: Continuous Time Case",to appear, *Proc. 1987 ACC*, Minneapolis, MN, June,1987.